1. Problem: Discrete-time random walk (10 points)

A drunkard moves along a line by making each second a step to the right or to the left with equal probability. Thus, his possible positions are the integers $-\infty < n < \infty$. We assume that initially $n = 0$ and want to determine the probability $p_n(r)$ after r steps.

1.1. Combinatorial derivation

Derive the probability $p_n(r)$ by combinatorial arguments.
Hint: How many different paths lead to the position n after r steps?

1.2. Solve by addition of variables

Each step $j = 1, 2, \ldots, r$ corresponds to a stochastic variable $\hat{x}_j$ taking on the values 1 and -1 with equal probability 1/2. The position after r steps is then given by

$$\hat{n}_r = \hat{x}_1 + \hat{x}_2 + \ldots + \hat{x}_r.$$

The steps and, thus, the variables $\hat{x}_j$ are mutually independent.

1.2.1. Average

Derive the average $\langle \hat{n}_r \rangle$ and the variance $\langle \big(\hat{n}_r - \langle \hat{n}_r \rangle\big)^2 \rangle$. Discuss the r-dependence of these variables and compare it to a deterministic (sober) movement.

1.2.2. Characteristic Function

Calculate the characteristic function $G_{\hat{n}_r}(k)$. Determine the probability $p_n(r)$ from this expression. What is the advantage of this method compared to the combinatorial derivation?

Hint:

$$(a+b)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^{n-k} b^k$$

1.3. Asymmetric random walk

Now assume that a step to the left has the probability q and to the right $1-q$. Find $p_n(r)$ for this case.

2. Problem: The Poisson distribution, cumulants (no points)

2.1. Poisson distribution

The Poisson distribution is defined as

$$p_n = \frac{a^n}{n!} \, \mathrm{e}^{-a} \tag{1}$$

on the discrete range $n = 0, 1, 2, ...$

The Poisson distribution describes, e.g., the number of events occuring in a certain time or spatial interval when the probability of an event occuring per time / spatial unit is constant. It also results from the binomial distribution in the limit of rare events ($\rightarrow$ 2.3).

Check the normalization of the Poisson distribution, i.e., $\sum_n p_n = 1$.

Calculate the first moment $\langle \hat{n} \rangle$.

Hint: Use the derivative $\frac{\partial}{\partial a}$ to bring the expression in a form which can readily be summed.

2.2. Cumulants

Calculate the cumulants κ_m of the Poisson distribution, Eq. (1), defined by

$$\log G(k) = \sum_{m=0}^{\infty} \frac{(\mathrm{i}k)^m}{m!} \kappa_m \, , \tag{2}$$

where $G(k) = \sum\limits_{n=0}^{\infty} p_n \mathrm{e}^{\mathrm{i}kn}$ is the characteristic function for a discrete random variable.
Hint: Use the derivative of $\log G(k)$.

2.3. Poisson theorem

Show that the Poisson distribution approximates the binomial distribution

$$p_n = \binom{N}{n} p^n \, q^{N-n} \,, \quad q = 1 - p \text{ and } n = 0, 1, 2, \ldots, N \tag{3}$$

in the limit $N \to \infty$, $p \to 0$, $Np = a = \text{const}$.

Hint: Use Stirling's formula $k! \approx \sqrt{2\pi k} \left(\frac{k}{\mathrm{e}} \right)^k$ for $k \to \infty$.

3. Probability generating function, factorial moments and cumulants (no points)

The *probability generating function* of a random variable $\hat{x}$ is defined by

$$f(z) = \left\langle z^{\hat{x}} \right\rangle \tag{4}$$

where z is a complex number on the unit circle $|z| = 1$.
What condition does $\hat{x}$ have to fulfil in order that $f(z)$ is even defined for all z within the unit circle? What could be the advantage of using $f(z)$ compared to the characteristic function $G(k) = \langle \mathrm{e}^{ik\hat{x}} \rangle$?

Calculate the probability generating function $f(z)$ for the Poisson distribution.

3.1. Factorial moments (optional)

For commonly occuring distributions, often special types of "moments", "cumulants" and corresponding generating functions have been introduced. They both facilitate calculations with these distributions, but are also sometimes used in approximation schemes. The "standard" moments and cumulants have already been introduced in the lecture. Here, we look at a variant which is sometimes helpful when dealing with Poisson distributions.

We first define the *factorial moments* Φ_m by

$$\Phi_0 = 1 \tag{5}$$

$$\Phi_m = \langle \hat{x}(\hat{x}-1)(\hat{x}-2)\ldots(\hat{x}-m+1) \rangle \quad (m \geq 1) \tag{6}$$

Show that the factorial moments are generated by the probability generating function:

$$f(1-y) = \sum_{m=0}^{\infty} \frac{(-y)^m}{m!} \Phi_m. \tag{7}$$

3.2. Factorial cumulants (optional)

The *factorial cumulants* θ_m are defined by

$$\log f(1-y) = \sum_{m=1}^{\infty} \frac{(-y)^m}{m!}\,\theta_m. \tag{8}$$

What is the relation between the factorial cumulants and the factorial moments up to θ_3.

How do the factorial moments look like?

What is special about the factorial cumulants of the Poisson distribution? Compare this to the case of Gaussian distributions.

4. Problem: Compound distribution (no points)

Let $\hat{x}_j$ be an infinite set of independent stochastic variables with identical distributions $P_{\hat{x}}(x)$ and characteristic function $G_{\hat{x}}(k) = \int\limits_{-\infty}^{\infty} P_{\hat{x}}(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x$. Let $\hat{r}$ be a random non-negative integer with distribution p_r and probability generating function $f(z)$. Then the sum

$$\hat{y} = \hat{x}_1 + \hat{x}_2 + \ldots + \hat{x}_{\hat{r}}$$

is a random variable, where we set $\hat{y} = 0$ for $\hat{r} = 0$. Show that its characteristic function $G_{\hat{y}}(k)$ fulfills

$$G_{\hat{y}}(k) = f(G_{\hat{x}}(k)).$$

The distribution of $\hat{y}$ is the so-called *compound distribution.*

4.1. Example

Assume that $\hat{x}_i$ are Bernoulli variables, i.e. $P(\hat{x}_i = 1) = p = 1 - q$ and $P(\hat{x}_i = -1) = q$ like in the asymetric random walk discussed in problem 1 and $\hat{r}$ obeys Poisson statistics. What is the characteristic function $G_{\hat{y}}(k)$ of the compound process? Use the characteristic function to compute both the mean and the variance of $\hat{y}$.