1. Problem: First-Passage problems and gambler's ruin (10 points)

In the following, we consider a one-dimensional discrete process described by the birth-death forward equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\, p(n,t|m,0) = \gamma^+_{n-1}\, p(n-1,t|m,0) + \gamma^-_{n+1}\, p(n+1,t|m,0) - \left(\gamma^+_n + \gamma^-_n\right)\, p(n,t|m,0) \qquad (1)$$

with the conditional probability[1] $p(n,t|m,0)$ for the process to be at time t at position $n \in \mathbb{Z}$ provided that initially, at time $t = 0$, we started at position $m \in \mathbb{Z}$. Obviously, this implies the initial condition

$$p(n, t=0|m,0) = \delta_{n,m}\,. \qquad (2)$$

The coefficient $\gamma^\pm_n$ is the probability per unit time to jump from n to $n \pm 1$.

Often, the dynamics (1) is restricted to a subset of all integers and thus boundary conditions come into play. For instance, one may be interested in the probability that the process reaches another site R (with, say, $R > m$) for the first time and, in case that this happens, after what time. This first-passage time is different for different realizations of the process and therefore a random quantity. In the following, we will derive relations for its average, the so-called *mean first-passage time*.[2]

A classical example of Eq. (1) with boundary conditions is the gambler's ruin problem: Suppose a gambler starts with an initial captial of m and the game continues until his capital is reduced to zero. How probable is that event and how long does it take on average until the game is finished? In the following, we will first study one possible approach for calculating these quantities and then consider a slightly more extended version of the gambler's ruin problem.

1.1. Absorbing boundary conditions

We assume that the site R is an absorbing boundary such that whenever a random walker hits R he is out. Thus Eq. (1) is only valid for $n < R-1$ and we have the additional equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\, p(R-1,t|m,0) = \gamma^+_{R-2}\;\, p(R-2,t|m,0) - \left(\gamma^+_{R-1} + \gamma^-_{R-1}\right) p(R-1,t|m,0)\,. \qquad (3)$$

[1] In order to clutter the notation not too much, we write $p(n,t|m,0)$ for $p_{1|1}(n,t|m,0)$.
[2] Mean-first-passage-time problems are also one of the most important topics of chemical physics: Reaction rates can be calculated as the inverse of the mean first-passage time in a continuous model for diffusion across a potential barrier.

The probability $p(R,t|m,0)$ does not appear anymore.

Please note that since transitions back to $R-1$ starting from R are not possible, the conservation of probability cannot be fulfilled:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=-\infty}^{R-1} p(n,t|m,0) = \gamma^{+}_{R-1}\, p(R-1,t|m,0)\,. \tag{4}$$

This can be interpreted as the density of an ensemble of independent particles each conducting a random walk until it disappears at $n = R$. The number of remaining particles decreases.[3] Note that both sides of Eq. (4) give the probability per unit time of disappearing at R at time t.

Show that the probability $\pi_R(m)$ of reaching R having started at m is given by

$$\pi_R(m) = \int_0^\infty \mathrm{d}t\,\left(-\sum_{n=-\infty}^{R-1}\frac{\mathrm{d}}{\mathrm{d}t}\,p(n,t|m,0)\right) \tag{5}$$

Show furthermore, provided that the process reaches R with probability one, i.e., $\pi_R(m) = 1$ that mean first-passage time $\tau_R(m)$ is given

$$\tau_R(m) = \int_0^\infty \mathrm{d}t\,t\left(-\sum_{n=-\infty}^{R-1}\frac{\mathrm{d}}{\mathrm{d}t}\,p(n,t|m,0)\right) = \sum_{n=-\infty}^{R-1}\int_0^\infty \mathrm{d}t\;p(n,t|m,0)\,. \tag{6}$$

Note that this requires the somewhat stronger assumption $\lim_{t\to\infty}\left[t\sum_{n=-\infty}^{R-1} p(n,t|m,0)\right] = 0$. Would you expect this to be fulfilled provided that $\pi_R(m) = 1$?

1.2. Backward equation for splitting probabilities and mean first-passage time

Many formalisms have been developed for the calculation of quantities relating to first-passage problems. Here, we shall derive one based on the backward equation, which for general, discrete time-homogeneous Markov process assumes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,p(n,t|m,0) = \sum_{m'}\Gamma^{\dagger}_{mm'}\,p(n,t|m',0) \tag{7}$$

"Derive" this from the backward equation for a continuous state space given in the lecture. How does the master operator Γ_{mm} and its adjunct $\Gamma^{\dagger}_{mm'}$ look for the birth-death process (1).

Derive for the splitting probability the equation

$$\sum_{m'}\Gamma^{\dagger}_{mm'}\,\pi_R(m') = 0 \tag{8}$$

[3]If you do not like this fact, introduce an absorbing state which captures the lost particles.

where $m < R$ with boundary condition $\pi_R(R) = 1$. Interpret the boundary condition. Rewrite this equation in the form

$$\pi_R(m) = \frac{\gamma_m^+}{\gamma_m^+ + \gamma_m^-}\,\pi_R(m+1) + \frac{\gamma_m^-}{\gamma_m^+ + \gamma_m^-}\,\pi_R(m-1)\,. \tag{9}$$

Similarly, show that for $\pi_R(m) = 1$ the mean first-passage time obeys

$$\sum_{m'} \Gamma^\dagger_{mm'}\,\tau_R(m') = -1 \tag{10}$$

with boundary condition $\tau_R(R) = 0$. Again, discuss the plausibility of the boundary condition.

1.3. Two absorbing boundaries

So far, we have only considered one absorbing boundary. To make the problem more interesting for the gambler (or the bank), we now allow for an alternative end of the game by introducing another boundary on the "other" side, say at a side $L < m$. Then we ask for the probabilities $\pi_R(m)$ and $\pi_L(m)$ that a process starting at m reaches first R or L, respectively. Thus, we introduce analogously to the equation at R, an absorbing boundary at L:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,p(L+1,t|m,0) = \gamma_{L+2}^-\,p(L+2,t|m,0) - \left(\gamma_{L+1}^+ + \gamma_{L+1}^-\right)\,p(L+1,t|m,0)\,. \tag{11}$$

The splitting probability $\pi_R(m)$ now still obeys the backward equation (9) with the same boundary condition at $m = R$ but additionally, it has to fulfill $\pi_R(L) = 0$. Why?

The concept of the mean first-passage time can be generalized to a mean exit-time $\tau(m)$ if we do not care about the fact where the process leaves the "allowed" range $L+1 < m < R-1$. Show that then (10) holds (replacing $\tau_R(m)$ by $\tau(m)$) with the boundary conditions $\tau(L) = \tau(R) = 0$.

1.4. Example: Gambler's ruin

Now we are finally able to consider an interesting version of the gambler's ruin problem, namely the probability that a gambler with initial capital m goes bancrupt (i.e., ends up with zero capital $L = 0$) if he plays against another gambler with capital $R-m$, when per unit time he wins (loses) one "coin" with probability γ_+ (γ_-). with asymmetrical chances for winning the game.[4]

[4]Note that somewhat unrealistically, we assume a continuous version of the game. It tunrs out, however, that the result for the more realistic discrete-time process is the same.

1.4.1. Ruin probability

The ruin probability is just given by the splitting probability $\pi_0(m)$ for $m > 0$ and we specialize the rates to $\gamma_m^+ = \gamma^+$ and $\gamma_m^- = \gamma^-$. Derive then the result

$$\pi_0(m) = \frac{x^R - x^m}{x^R - 1} \tag{12}$$

where $x = \gamma^-/\gamma^+$.

Hint: Consider the difference equation for $\Delta_m = \pi_0(m+1) - \pi_0(m)$. Use $\sum_{m=0}^{R-1} \Delta_m = -1$ (why?).

Discuss this in the limit $\gamma^- = \gamma^+$ and also for $R \to \infty$, i.e. the gambler playing against a bank.

1.4.2. Mean duration of game (no points)

Show that the mean duration of the game is given in the "symmetric" case $\gamma^+ = \gamma^- = 1$ by

$$\tau(m) = m(R - m)/2 \tag{13}$$

Study some numerical examples for this mean exit-time. Is the duration of the game longer or shorter than you expected it?