1. Problem: Dichotomic Noise or Random Telegraph Process (10 points)

We want to consider dichotomic noise, also known under the name random telegraph process:

$$\hat{x}(t) := a\,(-1)^{\hat{n}(t)}$$

i.e. a signal that has either value $+a$ or $-a$, where $a > 0$ is a constant and $\hat{n}(t)$ refers to a Poisson process with jump rate λ defined by

$$p_{1|1}(n_2, t_2|n_1, t_1) = \begin{cases} \dfrac{[\lambda(t_2 - t_1)]^{n_2 - n_1}}{(n_2 - n_1)!}\, \mathrm{e}^{-\lambda(t_2 - t_1)} & \text{for } n_2 \geq n_1 \\ 0 & \text{else} \end{cases}$$

$$p_1(n, t = 0) = \delta_{n,0}$$

where $t_1 \leq t_2$.

1.1. Mean, autocorrelation and spectral density

Calculate the mean value $\langle \hat{x}(t) \rangle$ and the autocorrelation function $\langle \hat{x}(t)\hat{x}(t') \rangle$ of this process.

Hint: Use $(-1)^{\hat{n}(t)}\,(-1)^{\hat{n}(t')} = (-1)^{\hat{n}(t) - \hat{n}(t')}$. Also consider $t \geq t'$ during the calculation and then deduce the result for $t < t'$.

Calculate the spectral density $S(\omega)$. In which limit does white noise emerge?

1.2. Probabilities

Due to the Markovian nature of the underlying Poisson process, the random telegraph process is also Markovian and, thus, fully described by the one-time probability and the two-time conditional probability. Derive the conditional probability $p_{1|1}(x_2, t_2|x_1, t_1)$ $(t_1 \leq t_2)$ and the probability $p_1(x, t)$ for this process.

Hint: Use the formula for the transformation of random variables given in the lecture.

1.3. Master equation

How does the master equation of this process look like? Interpret and discuss the result.

2. Problem: Wiener process (no points)

The Wiener process $\hat{W}(t)$ describes the asymptotic behavior of the symmetric random walk for $t \to \infty$, $n \to \infty$ and n^2/t fixed. The evolution of the continuous position variables is then given by

$$p_{1|1}(W_2, t_2|W_1, t_1) = \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left[-\frac{(W_2 - W_1)^2}{2(t_2 - t_1)}\right], \tag{1}$$

$$p_1(W, 0) = \delta(W) \tag{2}$$

where $t_1 \leq t_2$. The process $\hat{W}(t)$ is Gaussian, time-homogeneous but not stationary. It describes the diffusion of a Brownian particle. Another important feature of the Wiener process are its independent increments.

2.1. One-time probability

What is the one-time probability $p_1(W, t)$ for the Wiener process defined above?

2.2. Conditional expectation values

Show that for $0 < t_1 \leq t_2$

$$\left\langle \hat{W}(t_2) \mid W_1, t_1 \right\rangle = W_1,$$
$$\left\langle\!\left\langle \left[\hat{W}(t_2)\right]^2 \mid W_1, t_1 \right\rangle\!\right\rangle = t_2 - t_1,$$

where the expectation value and the second cumulant are conditioned on the value $W_1 = \hat{W}(t_1)$ of the process at the earlier time t_1: e.g. $\left\langle \hat{W}(t_2)|W_1, t_1 \right\rangle := \int \mathrm{d}W_2\, W_2\, p_{1|1}(W_2, t_2|W_1, t_1)$.

2.3. Correlations

Derive for $t_1, t_2, t_3, t_4 > 0$ the relations

$$\left\langle \hat{W}(t_1) \right\rangle = 0\,,$$
$$\left\langle \hat{W}(t_2)\hat{W}(t_1) \right\rangle = \min(t_1, t_2)\,,$$
$$\left\langle \left[\hat{W}(t_1) - \hat{W}(t_2)\right]\left[\hat{W}(t_3) - \hat{W}(t_4)\right] \right\rangle = |(t_1, t_2) \cap (t_3, t_4)|\,,$$

where the right-hand side of the third equation means the length of the overlap of both intervals. Interpret these results in connection with the property of independent increments that the Wiener process obeys.