The Hierarchical Nature of the Quantum Hall Effects

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Motivation

- The goal of this paper is to explicitly demonstrate the hierarchical nature of the IQH effect and of the CF theory of the FQH effect.
    - Application of the Haldane-Halperin hierarchy theory to the $\nu = n$ IQH states, to produce $\nu = n + \tilde{\nu}$ states, where $0 < \tilde{\nu} \leq 1$ is the filling of a quantum Hall state into which the quasielectrons are projected.

Hierarchy theory

Projection

In the hierarchy theory, one starts from a Wavefunction $\Psi(\mathbf{R}_\mu; \mathbf{r}_i)$ for a state of

- N electrons with coordinates $\mathbf{r}_1, \dots, \mathbf{r}_N$ that has
- N_{qp} quasiparticles at positions $\mathbf{R}_1, \dots, \mathbf{R}_{N_{qp}}$.

Then the quasiparticles are projected onto a quantum Hall type state with (pseudo-)wavefunction $\Phi(\mathbf{R}_\mu)$[1], giving the wavefunction for a new electronic state

$$\Psi'(\mathbf{r}_i) = \int \prod_{\mu=1}^{N_{qp}} d^2\mathbf{R}_\mu \, \overline{\Phi(\mathbf{R}_\mu)} \, \Psi(\mathbf{R}; \mathbf{r}_i). \tag{1}$$

[1] The wavefunction Φ must be suitably chosen so that the integrand is single-valued and hence well-defined.

Hierarchy theory

Interpretation

This hierarchy construction can be associated with the physical picture where an additional quantum Hall fluid is introduced and coupled to the original one by **bose condensing composites of quasiparticles** from the two Hall fluids that form electrically neutral bosons.

Such a topological bose condensation produces a **new topological phase** from the previous two.

Requiring that a **neutral boson composite** of quasiparticles can be formed provides precisely the same restriction on the allowed Hall fluid that may be added and hence its associated wavefunctions Φ, as does the single-valued integrand requirement.

Basis states

In spinor coordinates, the (single particle) orthonormal basis states are

$$\psi_{s,m}^{(j)}(\mathbf{r}) = \sqrt{\frac{(2s+1)(2s-j)!}{4\pi(s+m)!(s-m)!j!}}\mathcal{L}^j\left\{u^{s+m}v^{s-m}\right\} \qquad (3)$$

with

- Landau level j,
- eigenvalue of total angular momentum $s = \frac{1}{2}N_\phi + j$
- eigenvalue of L^z m
- the Landau level raising operator $\mathcal{L} = \bar{v}\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial v}$ and
- spherical coordinates $\mathbf{r}_i = (u_i, v_i)$.

Wavefunctions

Thus, a system filling the n lowest Landau levels $(j = 0, 1, \ldots, n-1)$ is described by the wavefunction

$$\chi_n(\mathbf{r}_i) = \sum_{\sigma \in S_N} \frac{(-1)^\sigma}{N!} \prod_{j=0}^{n-1} \chi_1^{(j)}\big(\mathbf{r}_{\sigma(\mathcal{N}_j + a_j)}\big) \tag{4}$$

where $\mathcal{N}_j = \sum_{r=0}^{j-1} N_r$, and $\chi_1^{(j)}$ is the wave function for a filled jth LL, given by

$$\chi_1^{(j)}(\mathbf{r}_a) = \sum_{\tau \in S_{N_j}} \frac{(-1)^\tau}{N_j!} \prod_{a=1}^{N_j} \psi^{(j)}_{s_j, s_j - a + 1}\big(\mathbf{r}_{\tau(a)}\big), \tag{5}$$

where $s_j = \frac{1}{2} N_\phi + j$.

Quasielectrons

In order to introduce N_{qe} quasielectrons in the $\nu = n$ state one decreases the flux by $\frac{1}{n}$ for each qe to $N_\phi = \frac{1}{n}N - n - \frac{1}{n}N_{qe}$. This **decreases the number of orbitals** in each LL and forces electrons to occupy LLs above the $(n-1)$th.
To form minimal uncertainty excitations that are maximally localized at specific points, one uses coherent states.

$$\begin{aligned}\phi^{(n)}_{s_n,\mathbf{R}}(\mathbf{r}) &= \frac{2s_n+1}{4\pi}\sqrt{\frac{(2s_n-n)!}{(2s_n)!n!}}\ \mathcal{L}^n\left\{(\bar{\alpha}u+\bar{\beta}v)^{2s_n}\right\} \\ &= \sum_{m=-s_n}^{s_n} \overline{\psi^{(0)}_{s_n,m}(\mathbf{R})}\ \psi^{(n)}_{s_n,m}(\mathbf{r})\end{aligned} \tag{6}$$

Quasielectrons

N electron wavefunction

Thus, the N electron wavefunction for this state with N_{qe} quasielectrons is given by

$$\begin{aligned}\Psi_n(\mathbf{R}_\mu;\mathbf{r}_i) = \sum_{\sigma\in S_N} \frac{(-1)^\sigma}{N!} \prod_{j=0}^{n-1} \chi_1^{(j)}\left(\mathbf{r}_{\sigma(\mathcal{N}_j+a_j)}\right) \\ \times \prod_{\mu=1}^{N_{qe}} \phi_{s_n,\mathbf{R}_\mu}^{(j)}\left(\mathbf{r}_{\sigma(\mathcal{N}_n+\mu)}\right)\end{aligned} \tag{7}$$

where this uses new values of N_j and s_j corresponding to $N_\phi = \frac{1}{n}N - n - \frac{1}{n}N_{qe}$.

Hierarchy theory

Projection II

Using $\overline{\Phi(\mathbf{R}_\mu)} = \tilde{\Psi}_{\tilde{\nu}}(\mathbf{R}_\mu)$, a 0th LL state with filling $0 < \nu \leq 1$ and shift $\tilde{S}$, the integral can only be non-zero if $s_n = \tilde{s}_0$, i.e. $N_\phi + 2n = \tilde{\nu}^{-1} N_{qe} - \tilde{S}$. This requires $N_{qe} = \frac{\tilde{\nu}}{n+\tilde{\nu}} N + \frac{n\tilde{\nu}(n+\tilde{S})}{n+\tilde{\nu}}$ and results in $N_\phi = \frac{1}{n+\tilde{\nu}} N - \frac{n^2+2n\tilde{\nu}+\tilde{\nu}\tilde{S}}{n+\tilde{\nu}}$ for the new state. By expanding $\tilde{\Psi}_{\tilde{\nu}}$ in terms of orbital occupation

$$\tilde{\Psi}_{\tilde{\nu}}(\mathbf{R}_\mu) = \sum_{m_1,\ldots,m_{N_{qe}}=-\tilde{s}_0}^{\tilde{s}_0} C_{[m_1,\ldots,m_{N_{qe}}]} \prod_{\mu=1}^{N_{qe}} \psi^{(0)}_{\tilde{s}_0,m_\mu}(\mathbf{R}_\mu) \tag{8}$$

where $C_{[m_1,\ldots,m_{N_{qe}}]}$ are the expansion coefficients, the integrals can be evaluated using the property

$$\int d^2\mathbf{R}\; \psi^{(0)}_{s_n,m}(\mathbf{R})\; \phi^{(n)}_{s_n,\mathbf{R}}(\mathbf{r}) = \psi^{(n)}_{s_n,m}(\mathbf{r}). \tag{9}$$

Wavefunction

The wavefunction resulting from using Eqs. (7) and (8) in the hierarchical construction for the new state is

$$\begin{aligned}\tilde{\Psi}'_{n+\tilde{\nu}}(\mathbf{r}_i) &= \int \prod_{\mu=1}^{N_{qe}} d^2\mathbf{R}_\mu \; \tilde{\Psi}_{\tilde{\nu}}(\mathbf{R}_\mu) \; \Psi_n(\mathbf{R}_\mu; \mathbf{r}_i) \\ &= \sum_{\sigma \in S_N} \frac{(-1)^\sigma}{N!} \prod_{j=0}^{n-1} \chi_1^{(j)}\left(\mathbf{r}_{\sigma(\mathcal{N}_j + a_j)}\right) \; \tilde{\Psi}_{\tilde{\nu}}^{(n)}\left(\mathbf{r}_{\sigma(\mathcal{N}_n + \mu)}\right)\end{aligned} \tag{10}$$

where

$$\tilde{\Psi}_{\tilde{\nu}}^{(n)}\left(\mathbf{r}_\mu\right) = \sum_{m_1, \ldots, m_{N_{qe}} = -s_n}^{s_n} C_{[m_1, \ldots, m_{N_{qe}}]} \prod_{\mu=1}^{N_{qe}} \psi_{s_n, m_\mu}^{(n)}\left(\mathbf{r}_\mu\right) \tag{11}$$

is the wavefunction obtained by raising $\tilde{\Psi}_{\tilde{\nu}}$ from the 0th LL to the nth LL (replacing s_0 with s_n).

IQH states posses a hierarchical structure

Projecting the quasielectrons onto a filled LL, i.e. using $\Phi(\mathbf{R}_\mu) = \overline{\chi_1(\mathbf{R}_\mu)}$ in Eq. (1) of the hierarchy construction, produces $\Psi'_{n+1}(\mathbf{r}_i) = \chi_{n+1}(\mathbf{r}_i)$, the wavefunction for the state with the $n+1$ lowest LLs filled.

The hierarchical structure of the IQH states is explicitly demonstrated.

Composite fermion theory

The CF theory generates $\nu = \frac{\nu^*}{2p\nu^* \pm 1}$ FQH states by viewing them as filling ν^* **quantum Hall states of CFs**, which are bound states of electrons with $\pm 2p$ quantized vortices.
The associated ground-state trial wavefunction generated from that of a ν^* state with wavefunction Ψ_{ν^*} is

$$\Psi_\nu(\mathbf{r}_i) = [\chi_1(\mathbf{r}_i)]^{2p} \Psi_{\pm\nu^*}(\mathbf{r}_i) \tag{12}$$

where $\Psi_{-\nu^*}(\mathbf{r}_i) = \overline{\Psi_{\nu^*}(\mathbf{r}_i)}$ corresponds to using negative vortices.
The experimentally prominent series of FQH states $\nu = \frac{n}{2pn \pm 1}$ corresponding to $\nu^* = n$ are thought of **IQH states of CFs**.
The other odd-denominator filling fractions are generated in the CF picture using $\nu^* = n + \tilde{\nu}$ with $0 < \tilde{\nu} \leq 1$.
Example: $\nu^* = 4/3$ gives $\nu = \frac{4}{8p \pm 3}$, which is relevant for the observed $\nu = 4/11$ state.

Conclusion

From this perspective, it is now obvious that the CF description of FQH states is hierarchical in nature, just as the IQH and $\nu = n + \tilde{\nu}$ FQH states are hierarchical in nature.

2nd part (not presented):
I also demonstrated the hierarchical structure of the CF FQH states directly through application of the hierarchy construction to Laughlin and CF trial wavefunctions, using quasielectron wavefunctions generated from the CF picture. One can now obtain the sought after phenomenology of the HH hierarchy theory for FQH states by simply adopting that of the CF theory.