Ramsey Numbers and Adiabatic Quantum Computing

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The graph-theoretic Ramsey numbers are notoriously difficult to calculate. In fact, for the two-color Ramsey numbers $R(m, n)$ with $m, n \geq 3$, only nine are currently known. We present a quantum algorithm for the computation of the Ramsey numbers $R(m, n)$. We show how the computation of $R(m, n)$ can be mapped to a combinatorial optimization problem whose solution can be found using adiabatic quantum evolution. We numerically simulate this adiabatic quantum algorithm and show that it correctly determines the Ramsey numbers $R(3, 3)$ and $R(2, s)$ for $5 \leq s \leq 7$. We then discuss the algorithm's experimental implementation, and close by showing that Ramsey number computation belongs to the quantum complexity class quantum Merlin Arthur.

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RAMSEY NUMBERS

Consider a group of N people.

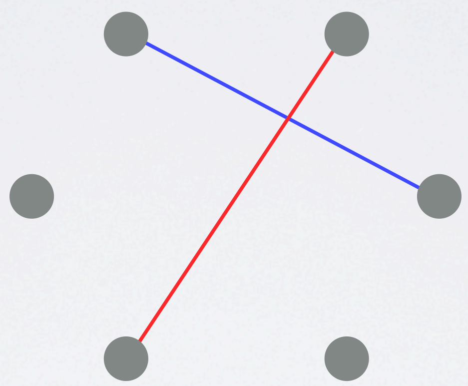
Ramsey's theorem states that:
There is a lower bound $R(n,m)$ such that when $N \geq R(n,m)$ all parties of N people will either contain m mutual acquaintances, or n mutual strangers.

$R(n,m)$ is a two-color Ramsey number.

Example: $R(2,2) = 2$

REPRESENTATION

Every person is represented by a vertex on a graph

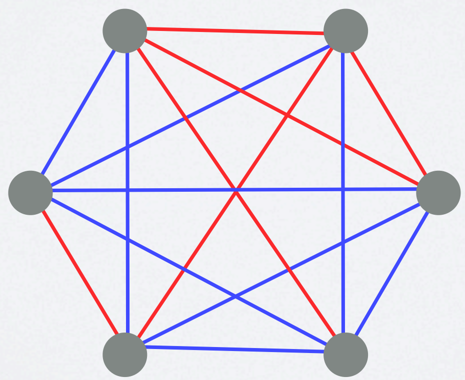


A red line denotes that two people are strangers

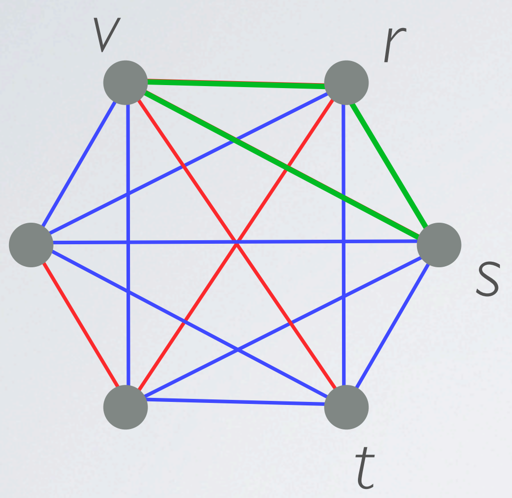
A blue line denotes that two people are acquainted

RAMSEY'S THEOREM IN OTHER WORDS

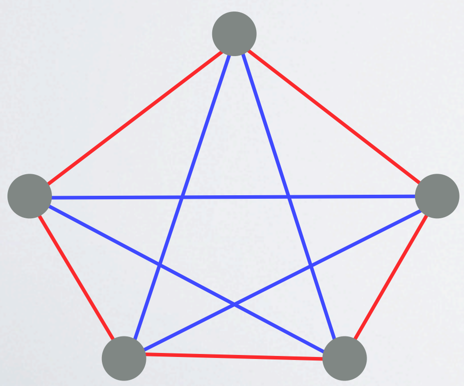
For every pair of positive integers (n,m) there exists a least positive integer $R(n,m)$ such that for any complete graph on $R(n,m)$ vertices, whose edges are colored red or blue, there exists either a complete subgraph on n vertices which is entirely blue, or a complete subgraph on m vertices which is entirely red.



EXAMPLE: $R(3,3) = 6$



- Pick a vertex *v*.
- Either ≥3 edges are red, or ≥3 blue.
- Assume red, connected to *r*,*s*,*t*.
- Either any of (*r*,*s*),(*s*,*t*),(*t*,*r*) is red, which gives a red triangle
- If they are all blue we have a blue triangle



One can make a graph with 5 vertices that has no closed triangles.

PROBLEM

Ramsey number are very hard to calculate.

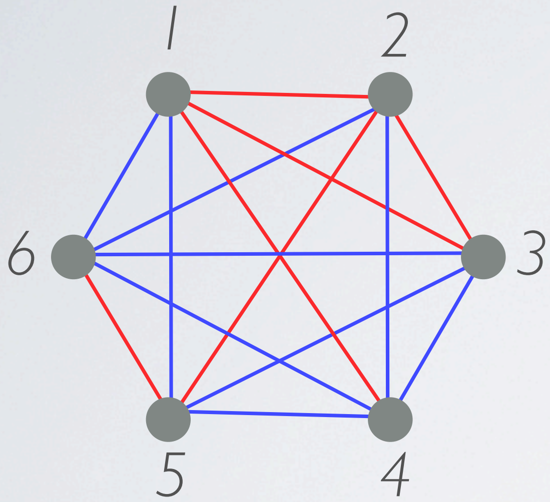
To check whether $N = R(n,m)$ you need to check all $2^{N(N-1)/2}$ N-vertex graphs.

This grows superexponentially.

For example:
$N = 2$ gives 2 graphs
$N = 5$ gives 1024 graphs
$N = 10$ gives 10^{13} graphs

MAPPING

The problem is mapped on a string of bits in 2 steps



$$\rightarrow \begin{pmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & & \\ 0 & 1 & 1 & 0 & & \\ 1 & 0 & 1 & 1 & 0 & \\ 1 & 1 & 1 & 1 & 0 & \end{pmatrix}$$

N(*N*-1)/2 unique elements

$$\begin{pmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & & \\ 0 & 1 & 1 & 0 & & \\ 1 & 0 & 1 & 1 & 0 & \\ 1 & 1 & 1 & 1 & 0 & \end{pmatrix} \rightarrow g(G) = a_{2,1} \ldots a_{N,1} a_{3,2} \ldots a_{N,2} \ldots a_{N,N-1}$$

Length is *N*(*N*-1)/2

OPTIMIZATION PROBLEM

Given a string g(G), how to determine # blue complete subgraph on *m* vertices for graph with *N* vertices?

Choose *m* vertices: $S_\alpha = \{v_1, \ldots, v_m\}$

Calculate the product: $C_\alpha = \Pi^{j \neq k}_{v_j, v_k \in S_\alpha} a_{v_j, v_k}$

If $C_\alpha = 1$ then S_α is a complete subgraph, otherwise not

Repeat this for all *r* possible ways of choosing *m* from *N*.

$C(G) = \sum_{\alpha=1}^{r} C_\alpha$ gives the # you are after.

OPTIMIZATION PROBLEM

Can do the same for red subgraphs

Choose *n* vertices: $T_\alpha = \{v_1, \ldots, v_n\}$

Calculate the product: $I_\alpha = \Pi^{j \neq k}_{v_j, v_k \in S_\alpha} \bar{a}_{v_j, v_k}$

If $I_\alpha = 1$ then T_α is a complete subgraph, otherwise not

Repeat this for all *s* possible ways of choosing *n* from *N*.

$I(G) = \sum_{\alpha=1}^{s} I_\alpha$ gives the # you are after.

OPTIMIZATION PROBLEM

Check all graphs *G* and find the G^* with the global minimum

$$h(G^*) = I(G^*) + C(G^*) \geq 0$$

If the global minimum is 0, there is a graph without complete subgraphs on *m* and *n* vertices.

Can determine R(m,n) by

- start from $N < R(n,m)$
- Increase N by 1 until you find an *N* for which $h(G^*) > 0$
- This $N = R(m,n)$

ADIABATIC QUANTUM COMPUTING

GS of H_p must solve the problem. H_i is a simple Hamiltonian

$$H(t) = \left(1 - \frac{t}{T}\right) H_i + \frac{t}{T} H_p$$

Use the computational basis

$$\sigma_z^0 \otimes \sigma_z^1 \otimes \cdots \otimes \sigma_z^{L-1}$$

Identify $g(G) \to |g(G)\rangle$ and take a Hamiltonian where

$$H_p|g(G)\rangle = h(G)|g(G)\rangle$$

Then increase N until $h(G) > 0$. This gives $R(m,n)$.

WHAT DOES H_P LOOK LIKE?

Given a set of vertices, $S_\alpha = \{v_1, \dots, v_m\}$ define the 'edge set'

$$E_\alpha = \{e_k^\alpha : k = 1, \dots, C(m, 2)\}$$

Define the Hamiltonian ($P_1^e = \frac{1}{2}(I^e - \sigma_z^e)$)

$$H_\alpha = \Pi_{e \in E_\alpha} P_1^e$$

Then the Hamiltonian $H^m = \sum_{\alpha=1}^{C(N,m)} H_\alpha$ counts C(G).

With $P_0^e = \frac{1}{2}(I^e + \sigma_z^e)$ one can make H^n which counts I(G)

WHAT DOES H_P LOOK LIKE?

Since $H_p|g(G)\rangle = h(G)|g(G)\rangle$ it follows that

$$H_p = H^n + H^m$$

So in principle the authors have shown a way to calculate Ramsey numbers using adiabatic quantum computation.

$R(2,5)$					$R(2,6)$					$R(3,3)$					$R(2,7)$				
N	E_{gs}	D	T	P_s	N	E_{gs}	D	T	P_s	N	E_{gs}	D	T	P_s	N	E_{gs}	D	T	P_s
3	0.0	1	5.0	0.591	4	0.0	1	5.0	0.349	4	0.0	18	5.0	0.769	5	0.0	1	8.0	0.865
4	0.0	1	5.0	0.349	5	0.0	1	5.0	0.173	5	0.0	12	5.0	0.194	6	0.0	1	8.0	0.805
5	1.0	11	5.0	0.518	6	1.0	16	5.0	0.286	6	2.0	1760	5.0	0.693	7	1.0	22	8.0	0.938

THE END