Quantum Time Crystals

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OUTLINE

Spontaneous symmetry breaking and time crystals

Particle on a ring and persistent current

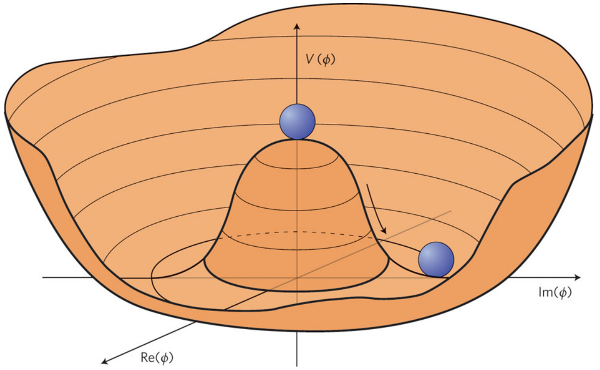
Soliton model and nonlinear Schrödinger equation

Discussion and controversy

Spontaneous symmetry breaking and time crystals

Hamiltonian has a symmetry, i.e.,

$$[H, U(\lambda)] = 0$$



Hamiltonian possesses many degenerate groundstates connected by the application of the symmetry operator

$$|\lambda\rangle = U(\lambda)|0\rangle$$ ➡ System spontaneously chooses one of the groundstates

Orthogonal states in thermodynamic limit

➡ Groundstate is less symmetric than the Hamiltonian

Examples:
- Real space crystals
- Mexican hat potential
- Heisenberg Ferromagnet
- etc…

What about time translation spontaneous symmetry breaking?

Spontaneous symmetry breaking and time crystals

Can time-translation be spontaneously broken ?

The authors answer is YES!

➡ **Time Crystals**

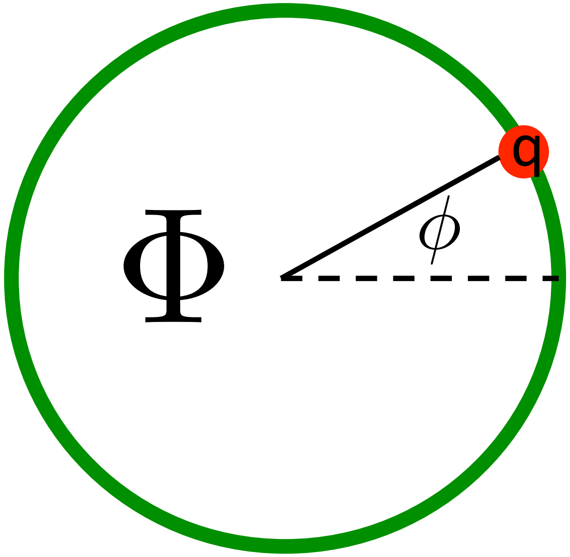
➡ Groundstate has periodic time dynamics

However, this seems to be impossible

$$\langle\Psi|\dot{\mathcal{O}}|\Psi\rangle = i\langle\Psi|[H,\mathcal{O}]|\Psi\rangle \xrightarrow[\Psi=\Psi_E]{} 0$$

Perpetuum mobile?

Particle on a ring and persistent current



Lagrangian $L = \frac{1}{2}\dot{\phi}^2 + \alpha\dot{\phi}$

Magnetic Flux $\Phi = 2\pi\alpha/q$

Momentum $\pi_\phi = \dot{\phi} + \alpha$

Hamiltonian $H = \pi_\phi\,\dot{\phi} - L = \frac{1}{2}(\pi_\phi - \alpha)^2$

➡ Eigenstates $|l\rangle = e^{il\phi}$

$$\langle l|\dot{\phi}|l\rangle = l - \alpha$$

$$\langle l|H|l\rangle = \frac{1}{2}(l - \alpha)^2$$

➡ $$\boxed{\langle l_0|\dot{\phi}|l_0\rangle = l_0 - \alpha}$$

Particle on a ring and persistent current

$$\boxed{\langle l_0|\dot{\phi}|l_0\rangle = \underset{\substack{\uparrow \\ \text{integer}}}{l_0} - \underset{\substack{\uparrow \\ \text{half-odd integer}}}{\alpha} \neq 0}$$

Not in disagreement with $\langle\Psi|\dot{\mathcal{O}}|\Psi\rangle = i\langle\Psi|[H,\mathcal{O}]|\Psi\rangle \xrightarrow[\Psi=\Psi_E]{} 0$

$\dot{\phi}$ is not a legitimate operator

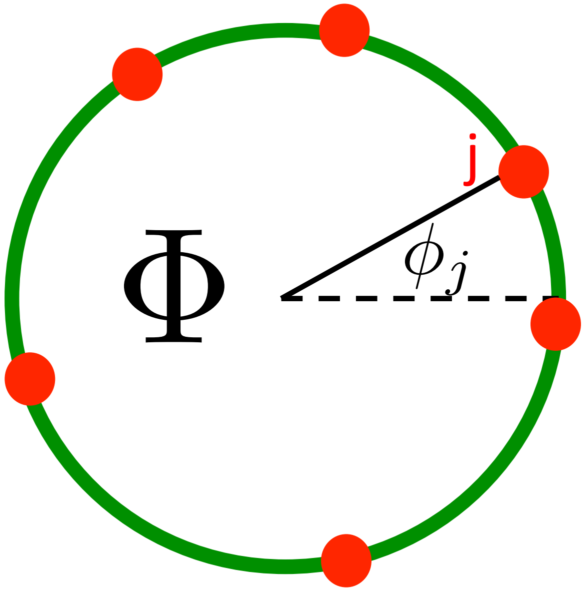
➡ This corresponds to wavefunction that appear in suoperconducting rings $\longrightarrow$ persistent current

If current is constant, then nothing changes in time!

➡ **Time translation symmetry is not broken**

Soliton model

Consider N particle on a ring



$$H = \sum_{j=1}^{N} \frac{1}{2}(\pi_j - \alpha)^2 - \frac{\lambda}{N-1} \sum_{j \neq k,1}^{N} \delta(\phi_j - \phi_k)$$

ultra-weak attractive contact interaction

Solution in mean-field approximation

➡ Product ansatz $\Psi(\phi_1, \ldots, \phi_n) = \prod_{j=1}^{N} \psi(\phi_j).$

Define effective potential V_{eff} such that $\langle \Psi | V_{\text{eff.}} | \Psi \rangle = \langle \Psi | V | \Psi \rangle$

$V_{\text{eff}}(\phi_1, \ldots, \phi_N) = \sum_{j=1}^{N} W(\phi_j)$ and $W(\phi_j) = \int \prod_{k \neq j} d\phi_k \psi^*(\phi_k) V \psi(\phi_k)$

Soliton model

$$i\frac{\partial\Psi}{\partial t} = \left(\sum_{j=1}^{N}\frac{1}{2}(\pi_j-\alpha)^2 + V_{\mathrm{eff.}}\right)\Psi$$

one-body nonlinear Schrödinger equation

$$\boxed{i\frac{\partial\psi}{\partial t} = \frac{1}{2}(\pi_\phi-\alpha)^2\psi - \lambda|\psi|^2\psi}$$

Solution without magnetic flux, i.e., α=0:

$$\psi(\phi,t) = e^{-i\mathcal{E}t}\psi_0(\phi+\beta)$$

$$\psi_0(\phi) = r\mathrm{dn}(r\sqrt{\lambda}\phi, k^2)$$

Jacobi elliptic function

$$\mathcal{E} = -r^2\lambda\left(1-\frac{k^2}{2}\right)$$

r and *k* are constants

Soliton model

Fix boundary conditions $\psi(\phi) = \psi(\phi + 2\pi)$

Normalization of $\psi_0(\phi)$

➡ Allow to determine constants *k* and *r*

$$E(k^2) = \frac{\sqrt{\lambda}}{2r} \qquad K(k^2) = \pi r\sqrt{\lambda}$$

elliptic integrals

➡ $E(k^2)K(k^2) = \frac{\pi\lambda}{2}$ can be solved

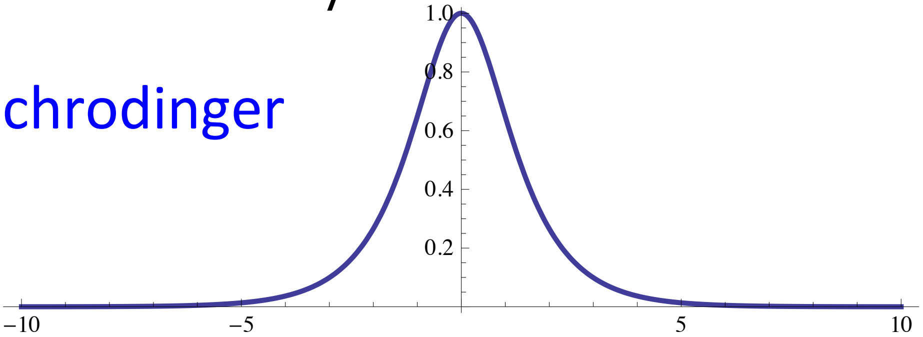
Minimum at *k=0* for *λ=π/2* and dn(u,0)=constant.

➡ energy $\mathcal{E} = -1/4$

Soliton model

$$\lambda > \pi/2 \quad \xrightarrow[k\text{->}1]{} \quad \mathrm{dn}(u, k^2) \longrightarrow \mathrm{sech}\, u$$

$$\mathcal{E} \longrightarrow -\lambda^2/8$$



Soliton solution of the nonlinear Schrodinger equation

$\boxed{\alpha \neq 0}$ nonzero magnetic flux

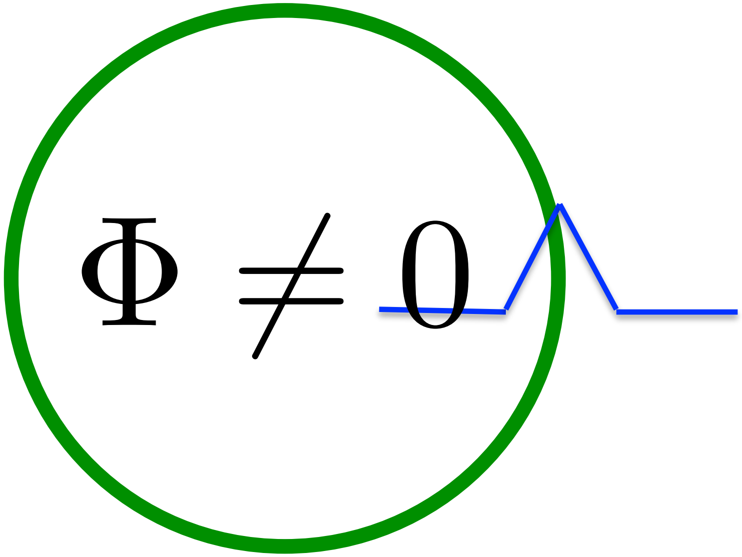
$$i\frac{\partial \psi_l}{\partial t} = \frac{1}{2}(-i\partial_\phi - \alpha)^2 \psi_l - \lambda |\psi_l|^2 \psi_l$$

$$\Rightarrow \quad \begin{aligned} \psi_l(\phi, t) &= e^{-il\phi}\, \tilde{\psi}(\phi + (l+\alpha)t, t) \\ i\frac{\partial \tilde{\psi}}{\partial t} &= \frac{1}{2}(-i\partial_\phi)^2 \tilde{\psi} - \lambda |\tilde{\psi}|^2 \tilde{\psi} + \frac{(l+\alpha)^2}{2} \tilde{\psi} \end{aligned}$$

$\Rightarrow$ $\boxed{\psi_{l_0}(\phi, t) \text{ is a moving soliton}}$

Soliton model

Intuitively



Faraday's Law

$$\Phi \neq 0$$

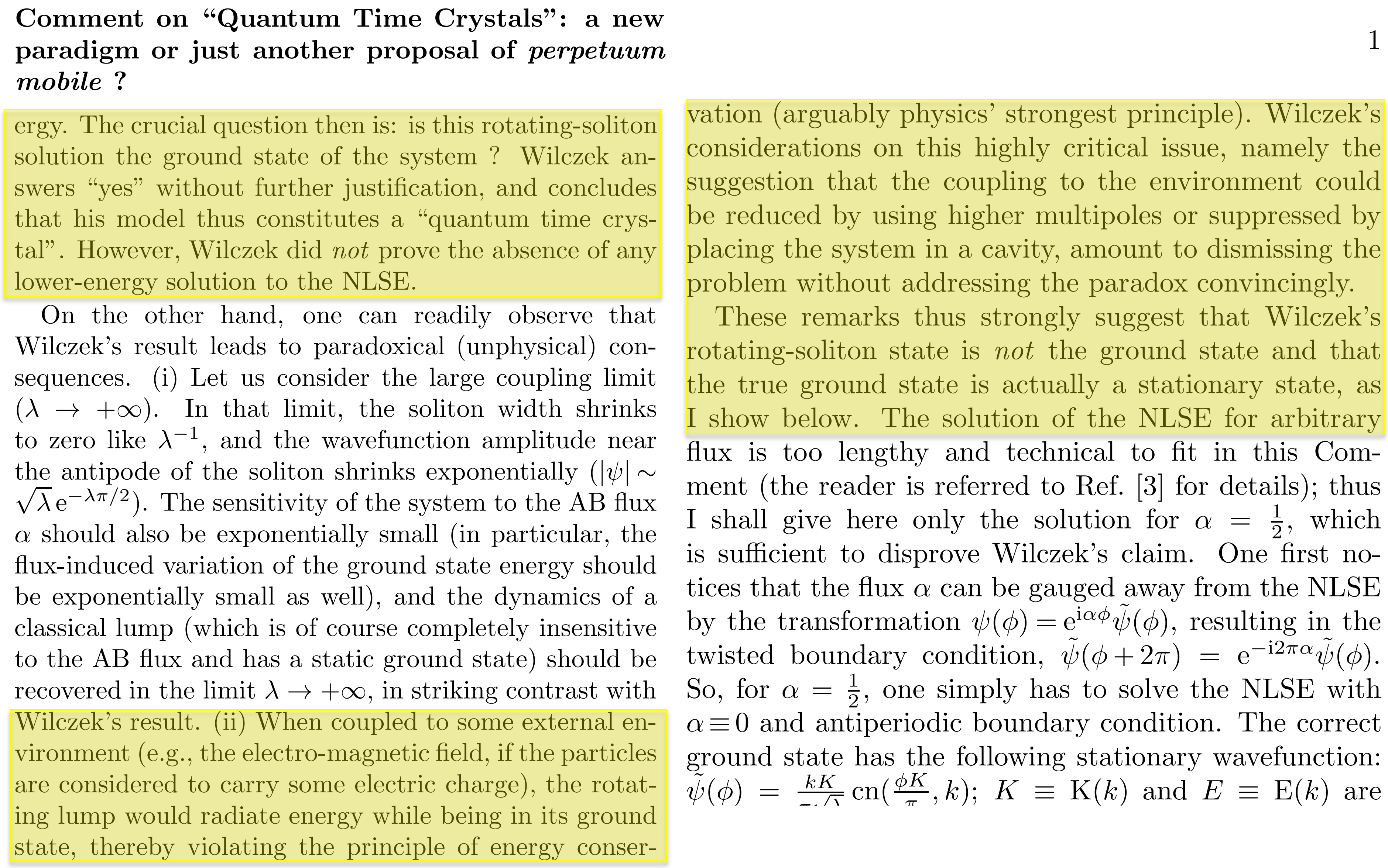
Break time-translation symmetry

Periodicity of the (inhomogeneous) soliton solution motion

➡ **Quantum time crystal**

Discussion and controversy

Comment by Patrick Bruno in Arxiv (15.10.2012)



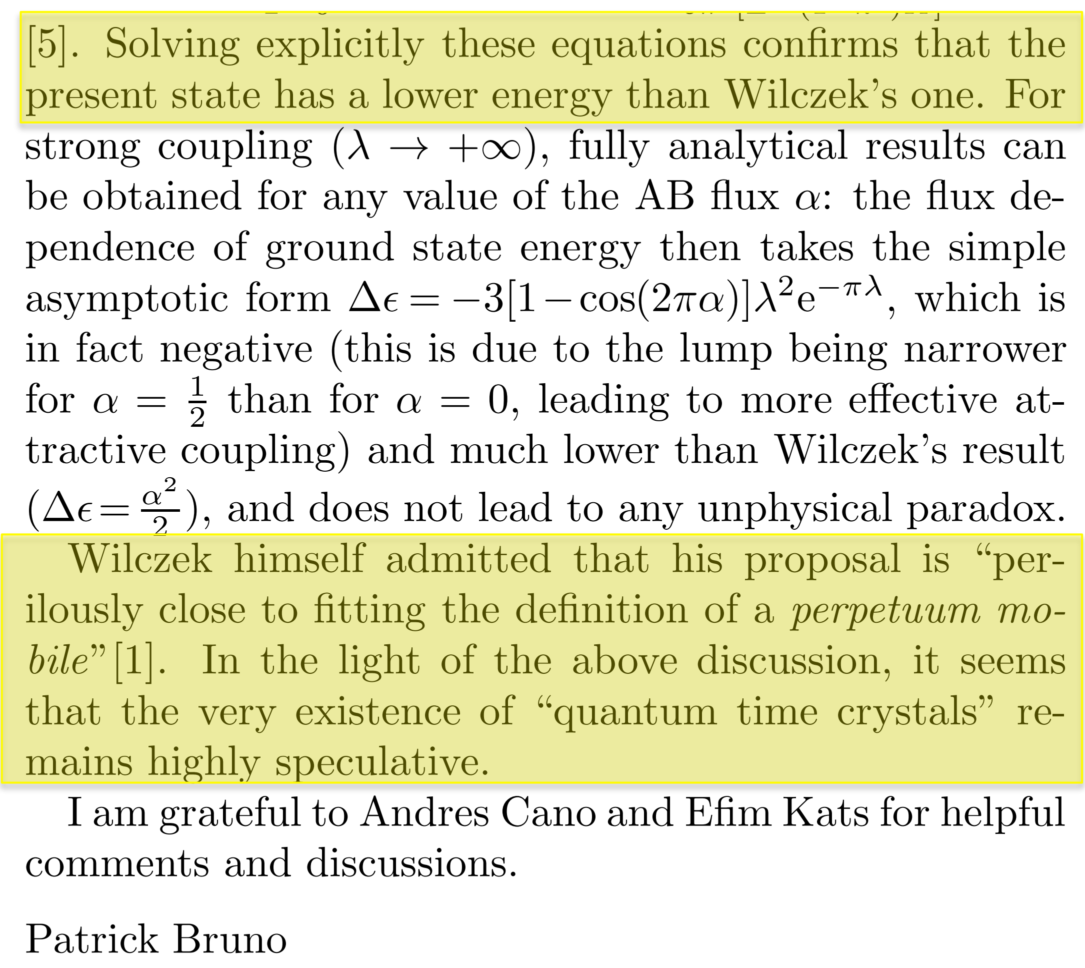
Comment on "Quantum Time Crystals": a new paradigm or just another proposal of *perpetuum mobile* ?

ergy. The crucial question then is: is this rotating-soliton solution the ground state of the system ? Wilczek answers "yes" without further justification, and concludes that his model thus constitutes a "quantum time crystal". However, Wilczek did *not* prove the absence of any lower-energy solution to the NLSE.

On the other hand, one can readily observe that Wilczek's result leads to paradoxical (unphysical) consequences. (i) Let us consider the large coupling limit ($\lambda \to +\infty$). In that limit, the soliton width shrinks to zero like λ^{-1}, and the wavefunction amplitude near the antipode of the soliton shrinks exponentially ($|\psi| \sim \sqrt{\lambda}\,\mathrm{e}^{-\lambda\pi/2}$). The sensitivity of the system to the AB flux α should also be exponentially small (in particular, the flux-induced variation of the ground state energy should be exponentially small as well), and the dynamics of a classical lump (which is of course completely insensitive to the AB flux and has a static ground state) should be recovered in the limit $\lambda \to +\infty$, in striking contrast with Wilczek's result. (ii) When coupled to some external environment (e.g., the electro-magnetic field, if the particles are considered to carry some electric charge), the rotating lump would radiate energy while being in its ground state, thereby violating the principle of energy conservation (arguably physics' strongest principle). Wilczek's considerations on this highly critical issue, namely the suggestion that the coupling to the environment could be reduced by using higher multipoles or suppressed by placing the system in a cavity, amount to dismissing the problem without addressing the paradox convincingly.

These remarks thus strongly suggest that Wilczek's rotating-soliton state is *not* the ground state and that the true ground state is actually a stationary state, as I show below. The solution of the NLSE for arbitrary flux is too lengthy and technical to fit in this Comment (the reader is referred to Ref. [3] for details); thus I shall give here only the solution for $\alpha = \frac{1}{2}$, which is sufficient to disprove Wilczek's claim. One first notices that the flux α can be gauged away from the NLSE by the transformation $\psi(\phi) = \mathrm{e}^{\mathrm{i}\alpha\phi}\tilde{\psi}(\phi)$, resulting in the twisted boundary condition, $\tilde{\psi}(\phi + 2\pi) = \mathrm{e}^{-\mathrm{i}2\pi\alpha}\tilde{\psi}(\phi)$. So, for $\alpha = \frac{1}{2}$, one simply has to solve the NLSE with $\alpha \equiv 0$ and antiperiodic boundary condition. The correct ground state has the following stationary wavefunction: $\tilde{\psi}(\phi) = \frac{kK}{\pi\sqrt{\lambda}}\,\mathrm{cn}(\frac{\phi K}{\pi}, k)$; $K \equiv \mathrm{K}(k)$ and $E \equiv \mathrm{E}(k)$ are

Discussion and controversy



[5]. Solving explicitly these equations confirms that the present state has a lower energy than Wilczek's one. For strong coupling ($\lambda \to +\infty$), fully analytical results can be obtained for any value of the AB flux α: the flux dependence of ground state energy then takes the simple asymptotic form $\Delta\epsilon = -3[1-\cos(2\pi\alpha)]\lambda^2 \mathrm{e}^{-\pi\lambda}$, which is in fact negative (this is due to the lump being narrower for $\alpha = \frac{1}{2}$ than for $\alpha = 0$, leading to more effective attractive coupling) and much lower than Wilczek's result ($\Delta\epsilon = \frac{\alpha^2}{2}$), and does not lead to any unphysical paradox.

Wilczek himself admitted that his proposal is "perilously close to fitting the definition of a *perpetuum mobile*"[1]. In the light of the above discussion, it seems that the very existence of "quantum time crystals" remains highly speculative.

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Patrick Bruno

THANK YOU