# Microscopic Model of Quasiparticle Wave Packets in Superfluids, Superconductors, and Paired Hall States

S. A. Parameswaran, S. A. Kivelson, R. Shankar, S. L. Sondhi, and B. Z. Spivak, Phys. Rev. Lett. 109, 237004 (2012)

Vladimir M. Stojanović @ Journal Club

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### What is this paper about?

#### • Quite general question:

How to think *in real space* about Bogoliubov quasiparticles in BCS (weakly-paired) superfluids, superconductors, or quantum Hall liquids? What kind of wave packet (or current pattern) represents such quasiparticles in real space?

#### • Answer provided in this paper:

Bogolons are represented by a dipolar current pattern!

⇒ Conclusion: "...Bogolons are fairly complicated objects!..."

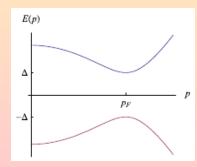
### BCS ground state and excitations: reminder

$$H_{ ext{BCS}} = \sum_{ ext{k},\sigma} \xi_{ ext{k}} c_{ ext{k},\sigma}^\dagger c_{ ext{k},\sigma} + \sum_{ ext{k}} \Delta_{ ext{k}} (c_{ ext{k},\uparrow}^\dagger \ c_{- ext{k},\downarrow} + ext{h.c.})$$

diagonalized:  $H_{ ext{BCS}} = \sum_{ ext{k},\sigma} E_{ ext{k}} \gamma_{ ext{k},\sigma}^{\dagger} \gamma_{ ext{k},\sigma} \; ; \quad E_{ ext{k}} = \sqrt{\xi_{ ext{k}}^2 + |\Delta_{ ext{k}}|^2}$ 

BCS ground state: 
$$|\Omega
angle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k},\uparrow}^{\dagger} \ c_{-\mathbf{k},\downarrow}^{\dagger}) |0
angle$$

$$egin{aligned} \gamma_{ ext{k},\uparrow} &= u_{ ext{k}} c_{ ext{k},\uparrow} &- v_{ ext{k}} c_{- ext{k},\downarrow}^{\dagger} \ \\ \gamma_{ ext{k},\downarrow} &= v_{ ext{k}} c_{ ext{k},\uparrow}^{\dagger} &+ u_{ ext{k}} c_{- ext{k},\downarrow} \ \\ |u_{ ext{k}}|^2 &= rac{1}{2} \left( 1 + rac{\xi_{ ext{k}}}{E_{ ext{k}}} 
ight) ; |v_{ ext{k}}|^2 = rac{1}{2} \left( 1 - rac{\xi_{ ext{k}}}{E_{ ext{k}}} 
ight) \end{aligned}$$



### Bogolon wave packet: a paradox

average momentum  ${f k}_0=k_{\scriptscriptstyle F} {f {\hat k}}_0$  and spatial extent  $\lambda~(\lambda\gg v_{\scriptscriptstyle F}/\Delta_0)$ 

$$|\Psi_{\mathrm{k}_0,s}^{\pmb{\lambda}}
angle = \left(rac{\pmb{\lambda}}{\sqrt{\pi}}
ight)^{d/2} \int d^d k \ e^{-(1/2) \pmb{\lambda}^2 (\mathrm{k}-\mathrm{k}_0)^2} |\mathrm{k},s
angle$$

quasiparticle current operator:  $\mathbf{j}^{\mathrm{qp}}_{\mathrm{q}} = \sum_{\mathrm{k},\sigma} \frac{\mathrm{k}}{m} \, c^{\dagger}_{\mathrm{k+q/2},\sigma} c_{\mathrm{k-q/2},\sigma}$ 

$$\langle {
m j}_{
m q}^{
m qp}
angle = v_{\scriptscriptstyle F} {
m \widehat{k}}_{\scriptscriptstyle 0} e^{-(\lambda^2 {
m q}^2)/4}$$

nonzero divergence for a stationary wave packet ⇒ violation of the continuity equation!!

treating the pair potential  $\Delta$  as homogeneous is inadequate!  $\Rightarrow$  self-consistent treatment is necessary

### Hubbard-Stratonovich (HS) transformation: a reminder

quartic term: 
$$V_{\alpha\beta\gamma\delta}\psi_{\alpha}^*\psi_{\beta}\psi_{\gamma}^*\psi_{\delta} = V_{\alpha\beta\gamma\delta}\underbrace{\rho_{\alpha\beta}}_{\rho_m}\underbrace{\rho_{\gamma\delta}}_{\rho_n} = \rho_m V_{mn}\rho_n$$

$$e^{-\rho_m V_{mn}\rho_n} = \mathcal{N}\int \mathcal{D}\phi\ e^{-\frac{1}{4}\phi_m (V^{-1})_{mn}\phi_n - i\phi_m\rho_n}$$

$$\mathcal{L}_0 = \sum_{\sigma=\uparrow} \psi_\sigma^* \left( i \partial_t - rac{
abla^2}{2m} - \mu 
ight) \psi_\sigma + g \psi_\uparrow^* \; \psi_\downarrow^* \psi_\downarrow \psi_\uparrow \; \; (g < 0)$$

HS transformation (Cooper channel) with the auxiliary pair-field  $\Delta(x)$ :

$$ilde{\mathcal{L}} = \psi_\sigma^* \left( i \partial_t - rac{
abla^2}{2m} - \mu 
ight) \psi_\sigma + \left[ \psi_\uparrow^* \; \psi_\downarrow^* \Delta(x) + ext{c.c.} 
ight] + rac{1}{g} |\Delta(x)|^2 \, .$$

broken-symm. phase – neglect amplitude fluctuations:  $\Delta(x) = \Delta e^{i heta(x)}$ 

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# Effective field-theory approach

$$\mathcal{L}_0 = \sum_{\sigma=\uparrow \; ,\downarrow} \psi_\sigma^* \left( i \partial_t - \mu - rac{
abla^2}{2m} 
ight) \psi_\sigma$$

$$\mathcal{L}_p = -\Delta_0 e^{i\theta(\mathbf{r},t)} \psi_{\uparrow}^* (\mathbf{r},t) \psi_{\downarrow}^* (\mathbf{r},t) + \text{h.c.}$$

$$\mathcal{L}_{ heta} = -rac{\kappa_0}{2}(\partial_t heta)^2 + rac{n_s}{2m}(
abla heta)^2$$

total density:  $ho=
ho^{
m qp}-\kappa_0\partial_t heta$  total current:  ${f j}={f j}^{
m qp}+rac{n_s}{2m}
abla heta$ 

equations of motion for the action  $S=\int dt d^dr (\mathcal{L}_0+\mathcal{L}_p+\mathcal{L}_ heta)$  :

$$oxed{ eta_t 
ho^{ ext{ iny qp}} = -
abla \cdot \mathbf{j}^{ ext{ iny qp}} + \mathcal{B}_p } egin{aligned} \kappa_0 \partial_t^2 heta = rac{n_s}{2m} 
abla^2 heta + \mathcal{B}_p \end{aligned}$$

$${\cal B}_p = 2i\Delta_0(e^{i\theta}\psi_\uparrow^*\;\psi_\downarrow^* - e^{-i\theta}\psi_\downarrow\psi_\uparrow$$
 ) continuity eq.:  $\partial_t \rho + 
abla \cdot {f j} = 0$ 

## Stationary Bogolon wave packet

$$\langle 
abla \cdot {
m j}^{
m qp} 
angle = \langle {\cal B}_p 
angle = rac{n_s}{2m} \langle 
abla^2 heta 
angle \ \Rightarrow {
m for} \; \langle {
m j}^{
m qp}_{
m qp} 
angle = v_F {
m k}_0 e^{-(\lambda^2 {
m q}^2)/4}$$

phase texture:

$$\langle heta_{
m q} 
angle_\Psi = rac{i({
m q} \cdot \widehat{
m k}_{
m 0})}{q^2} \left(rac{2k_F}{n_s}
ight) e^{-(\lambda^2 {
m q}^2)/4}$$

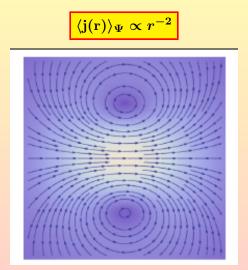
total current:

$$\langle \mathbf{j}_{\mathrm{q}} 
angle_{\Psi} = v_{\scriptscriptstyle F} \left[ rac{q^2 \widehat{\mathrm{k}}_{\scriptscriptstyle 0} - (\mathrm{q} \cdot \widehat{\mathrm{k}}_{\scriptscriptstyle 0}) \mathrm{q}}{q^2} 
ight] e^{-(\lambda^2 \mathrm{q}^2)/4}$$

current pattern in real space:

$$egin{aligned} \langle \mathbf{j}(\mathbf{r}) 
angle_{\Psi} &= \hat{z} imes 
abla \phi_{\lambda}(\mathbf{r}) \end{aligned} \Rightarrow ext{ solenoidal flow: } 
abla \cdot \langle \mathbf{j}(\mathbf{r}) 
angle_{\Psi} &= 0 \ \\ \phi_{\lambda}(\mathbf{r}) &= 2\pi v_F \frac{(\hat{\mathbf{k}}_0 imes \mathbf{r}) \cdot \hat{z}}{r^2} (1 - e^{-r^2/\lambda^2}) \end{aligned}$$

### Dipolar current pattern



as usual, corrections at short distances are nonuniversal

### **Superconductors**

$$egin{aligned} \partial_{\mu} 
ightarrow D_{\mu} &= \partial_{\mu} - i A_{\mu} & \mathcal{L}_{ ext{Maxwell}} &= rac{1}{4} F^{\mu 
u} F_{\mu 
u} \ \end{aligned} \ ext{density: } 
ho = 
ho^{ ext{qp}} - \kappa_0 (\partial_t heta - 2 A_0) & ext{current: } \mathbf{j} &= \mathbf{j}^{ ext{qp}} + rac{n_s}{2m} (
abla heta - 2 \mathbf{A}) \end{aligned}$$

total action  $S + S_{\mathrm{Maxwell}} \Rightarrow$  equations of motion:

extended Bogolon states here do not carry current (cancelled by a superfluid backflow), reflecting the Meissner effect:

$$\langle {
m j}
angle = 0 \Rightarrow$$
 if  $heta = 0$  (unitary gauge),  $\dfrac{n_s}{m}{
m A} = \langle {
m j}^{
m qp}
angle = v_F {
m \hat k}_0$ 

# Wave packet in the superconductor case

static wave packet 
$$(\partial_t 
ho^{ ext{qp}} = 0) \Rightarrow rac{n_s}{m} 
abla \cdot \mathbf{A} = \langle \mathcal{B}_p 
angle = \langle 
abla \cdot \mathbf{j}^{ ext{qp}} 
angle$$

Maxwell eq.  $\nabla imes \mathbf{B} = 4\pi \mathbf{j} \quad \Rightarrow$ 

$$oxed{[-
abla^2+\lambda_{
m L}^{-2}]{
m A}=4\pi\langle {
m j}^{
m qp}-\lambda_{
m L}^2
abla(
abla\cdot{
m j}^{
m qp})
angle}$$

penetration depth:  $\lambda_{
m L}^{-2}=4\pi rac{n_s}{m}$ 

$$oxed{\langle \mathbf{j}_{\mathrm{q}} 
angle_{\Psi} = v_F \left[ rac{q^2 \widehat{\mathbf{k}}_{\mathrm{0}} - (\mathbf{q} \cdot \widehat{\mathbf{k}}_{\mathrm{0}}) \mathbf{q}}{q^2 + \lambda_{\mathrm{L}}^{-2}} 
ight] e^{-(\lambda^2 \mathbf{q}^2)/4}}$$

exponential decay at long distances:

$$\langle {
m j}({
m r})
angle_\Psi \propto e^{-r/\lambda_{
m L}} ~~(r\gg \lambda,\lambda_{
m L})$$

### Paired Quantum Hall state

fermions in a (static) uniform background field A ( $B = \nabla \times A$ ) Chern-Simons term:  $\mathcal{L}_{\rm CS} = (4\Phi_0)^{-1} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$ 

density:  $ho = 
ho^{ ext{qp}} - \kappa_0 [\partial_t \theta - 2(a_0 + A_0)]$  current:  $\mathbf{j} = \mathbf{j}^{ ext{qp}} + rac{n_s}{2m} [
abla heta - 2(a + A)]$ 

total action  $S+S_{\mathrm{CS}}\Rightarrow$  equations of motion: change  $A_0 \rightarrow a_0 + A_0$ ,

 $\mathbf{A} 
ightarrow \mathbf{a} + \mathbf{A}$  from the "superconductor" case

$$\partial_t 
ho^{ ext{qp}} = - 
abla \cdot \mathrm{j}^{ ext{qp}} + \mathcal{B}_p$$

$$\kappa_0 \partial_t [\partial_t heta - 2(a_0 + A_0)] = rac{n_s}{2m} 
abla \cdot [
abla heta - 2(a + A)] + \mathcal{B}_p$$

$$[-
abla^2 + \lambda_{ ext{CS}}^{-2}]( ext{a} + ext{A}) = 8\kappa_0 \Phi_0^2 \langle ext{j}^{ ext{qp}} - \lambda_{ ext{CS}}^2 
abla (
abla \cdot ext{j}^{ ext{qp}}) 
angle$$

exponential decay with screening length:  $\lambda_{
m CS}^{-2}=8\kappa_0\Phi_0^2rac{n_s}{m}$ 

dipolar charge distribution stemming from  $\mathbf{e} = -\partial_t a_0 - \nabla \mathbf{a} = 2\Phi_0 \hat{z} imes \mathbf{j}$ 

#### Some remarks

- the results valid provided  $ar{
  ho}^{ ext{\tiny qp}} \xi^2 \ll 1$  (distance between quasiparticles is much larger than their size)
- Galilean-invariant system:

$${f J}({f x})=rac{e}{m}{f P}({f x})\Longrightarrow {f total} {f current} {f is} {f \it ev}_F$$
 [C. Nayak  ${\it et al.}$ , PRB  ${f 64}$ , 235113 (2001)]

 weak coupling theory in momentum space, vs. gauge-invariant strong-coupling case...