Microscopic Model of Quasiparticle Wave Packets in Superfluids, Superconductors, and Paired Hall States

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What is this paper about?

- **Quite general question:**

  How to think *in real space* about Bogoliubov quasiparticles in BCS (weakly-paired) superfluids, superconductors, or quantum Hall liquids?

  What kind of wave packet (or current pattern) represents such quasiparticles in real space?

- **Answer provided in this paper:**

  Bogolons are represented by a dipolar current pattern!

  $\Longrightarrow$ Conclusion: "...Bogolons are fairly complicated objects!..."

BCS ground state and excitations: reminder

$$H_{\mathrm{BCS}} = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} + \sum_{\mathbf{k}} \Delta_{\mathbf{k}} (c^{\dagger}_{\mathbf{k},\uparrow}\ c_{-\mathbf{k},\downarrow} + \mathrm{h.c.})$$

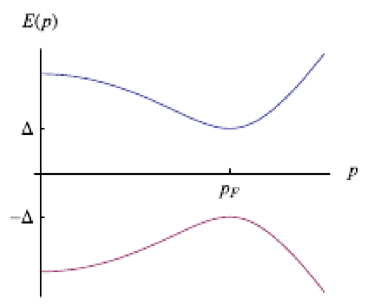
diagonalized: $$H_{\mathrm{BCS}} = \sum_{\mathbf{k},\sigma} E_{\mathbf{k}} \gamma^{\dagger}_{\mathbf{k},\sigma} \gamma_{\mathbf{k},\sigma} \quad ; \quad E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$$

BCS ground state: $$|\Omega\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^{\dagger}_{\mathbf{k},\uparrow}\ c^{\dagger}_{-\mathbf{k},\downarrow})|0\rangle$$

$$\gamma_{\mathbf{k},\uparrow} = u_{\mathbf{k}} c_{\mathbf{k},\uparrow} - v_{\mathbf{k}} c^{\dagger}_{-\mathbf{k},\downarrow}$$

$$\gamma_{\mathbf{k},\downarrow} = v_{\mathbf{k}} c^{\dagger}_{\mathbf{k},\uparrow} + u_{\mathbf{k}} c_{-\mathbf{k},\downarrow}$$

$|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$; $|v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$



Bogolon wave packet: a paradox

average momentum $\mathbf{k}_0 = k_F \hat{\mathbf{k}}_0$ and spatial extent λ ($\lambda \gg v_F/\Delta_0$)

$$|\Psi^{\lambda}_{\mathbf{k}_0,s}\rangle = \left(\frac{\lambda}{\sqrt{\pi}}\right)^{d/2} \int d^d k \, e^{-(1/2)\lambda^2(\mathbf{k}-\mathbf{k}_0)^2} |\mathbf{k}, s\rangle$$

quasiparticle current operator: $\mathbf{j}^{\mathrm{qp}}_{\mathbf{q}} = \sum_{\mathbf{k},\sigma} \frac{\mathbf{k}}{m} c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,\sigma} c_{\mathbf{k}-\mathbf{q}/2,\sigma}$

$$\langle \mathbf{j}^{\mathrm{qp}}_{\mathbf{q}} \rangle = v_F \hat{\mathbf{k}}_0 e^{-(\lambda^2 \mathbf{q}^2)/4}$$

nonzero divergence for a stationary wave packet
$\Rightarrow$ violation of the continuity equation!!

treating the pair potential Δ as homogeneous is inadequate!
$\Rightarrow$ self-consistent treatment is necessary

Hubbard-Stratonovich (HS) transformation: a reminder

quartic term: $V_{\alpha\beta\gamma\delta}\psi^*_\alpha\psi_\beta\psi^*_\gamma\psi_\delta = V_{\alpha\beta\gamma\delta}\underbrace{\rho_{\alpha\beta}}_{\rho_m}\underbrace{\rho_{\gamma\delta}}_{\rho_n} = \rho_m V_{mn}\rho_n$

$$e^{-\rho_m V_{mn}\rho_n} = \mathcal{N}\int \mathcal{D}\phi\, e^{-\frac{1}{4}\phi_m (V^{-1})_{mn}\phi_n - i\phi_m\rho_n}$$

$$\mathcal{L}_0 = \sum_{\sigma=\uparrow,\downarrow}\psi^*_\sigma\left(i\partial_t - \frac{\nabla^2}{2m} - \mu\right)\psi_\sigma + g\psi^*_\uparrow\ \psi^*_\downarrow\psi_\downarrow\psi_\uparrow \quad (g<0)$$

HS transformation (Cooper channel) with the auxiliary pair-field $\Delta(x)$:

$$\tilde{\mathcal{L}} = \psi^*_\sigma\left(i\partial_t - \frac{\nabla^2}{2m} - \mu\right)\psi_\sigma + [\psi^*_\uparrow\ \psi^*_\downarrow\Delta(x) + \mathrm{c.c}] + \frac{1}{g}|\Delta(x)|^2\,.$$

broken-symm. phase – neglect amplitude fluctuations: $\Delta(x) = \Delta e^{i\theta(x)}$

Effective field-theory approach

$$\mathcal{L}_0 = \sum_{\sigma=\uparrow,\downarrow} \psi^*_\sigma \left(i\partial_t - \mu - \frac{\nabla^2}{2m} \right) \psi_\sigma$$

$$\mathcal{L}_p = -\Delta_0 e^{i\theta(\mathrm{r},t)} \psi^*_\uparrow(\mathbf{r},t) \psi^*_\downarrow(\mathbf{r},t) + \mathrm{h.c.}$$

$$\mathcal{L}_\theta = -\frac{\kappa_0}{2}(\partial_t\theta)^2 + \frac{n_s}{2m}(\nabla\theta)^2$$

total density: $\rho = \rho^{\mathrm{qp}} - \kappa_0 \partial_t \theta$ total current: $\mathbf{j} = \mathbf{j}^{\mathrm{qp}} + \frac{n_s}{2m}\nabla\theta$

equations of motion for the action $S = \int dt d^d r (\mathcal{L}_0 + \mathcal{L}_p + \mathcal{L}_\theta)$:

$$\partial_t \rho^{\mathrm{qp}} = -\nabla \cdot \mathbf{j}^{\mathrm{qp}} + \mathcal{B}_p \qquad \kappa_0 \partial_t^2 \theta = \frac{n_s}{2m}\nabla^2\theta + \mathcal{B}_p$$

$\mathcal{B}_p = 2i\Delta_0 (e^{i\theta}\psi^*_\uparrow \psi^*_\downarrow - e^{-i\theta}\psi_\downarrow\psi_\uparrow)$ continuity eq.: $\partial_t\rho + \nabla\cdot\mathbf{j} = 0$

Stationary Bogolon wave packet

$$\langle \nabla \cdot \mathbf{j}^{\mathrm{qp}} \rangle = \langle \mathcal{B}_p \rangle = \frac{n_s}{2m} \langle \nabla^2 \theta \rangle$$

$\Rightarrow$ for $\langle \mathbf{j}^{\mathrm{qp}}_{\mathbf{q}} \rangle = v_F \widehat{\mathbf{k}}_0 e^{-(\lambda^2 \mathbf{q}^2)/4}$

phase texture:

$$\langle \theta_{\mathbf{q}} \rangle_{\Psi} = \frac{i(\mathbf{q} \cdot \widehat{\mathbf{k}}_0)}{q^2} \left(\frac{2k_F}{n_s} \right) e^{-(\lambda^2 \mathbf{q}^2)/4}$$

total current:

$$\langle \mathbf{j}_{\mathbf{q}} \rangle_{\Psi} = v_F \left[\frac{q^2 \widehat{\mathbf{k}}_0 - (\mathbf{q} \cdot \widehat{\mathbf{k}}_0)\mathbf{q}}{q^2} \right] e^{-(\lambda^2 \mathbf{q}^2)/4}$$

current pattern in real space:

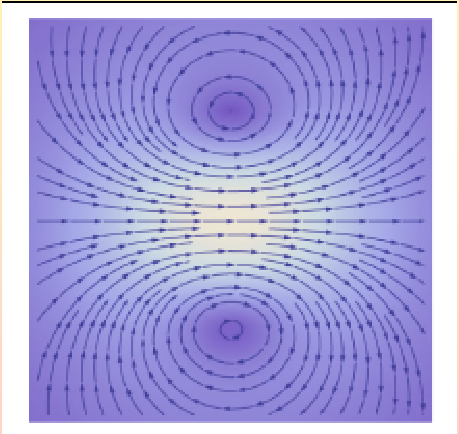
$$\langle \mathbf{j}(\mathbf{r}) \rangle_{\Psi} = \hat{z} \times \nabla \phi_{\lambda}(\mathbf{r})$$

$\Rightarrow$ solenoidal flow: $\nabla \cdot \langle \mathbf{j}(\mathbf{r}) \rangle_{\Psi} = 0$

$$\phi_{\lambda}(\mathbf{r}) = 2\pi v_F \frac{(\widehat{\mathbf{k}}_0 \times \mathbf{r}) \cdot \hat{z}}{r^2} (1 - e^{-r^2/\lambda^2})$$

Dipolar current pattern

$$\langle \mathbf{j}(\mathbf{r}) \rangle_{\Psi} \propto r^{-2}$$



as usual, corrections at short distances are nonuniversal

Superconductors

$$\partial_\mu \to D_\mu = \partial_\mu - iA_\mu \qquad \mathcal{L}_{\text{Maxwell}} = \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

density: $\rho = \rho^{\text{qp}} - \kappa_0(\partial_t\theta - 2A_0)$ current: $\mathbf{j} = \mathbf{j}^{\text{qp}} + \frac{n_s}{2m}(\nabla\theta - 2\mathbf{A})$

total action $S + S_{\text{Maxwell}} \Rightarrow$ equations of motion:

$$\partial_t \rho^{\text{qp}} = -\nabla \cdot \mathbf{j}^{\text{qp}} + \mathcal{B}_p$$

$$\kappa_0 \partial_t(\partial_t\theta - 2A_0) = \frac{n_s}{2m}\nabla \cdot (\nabla\theta - 2\mathbf{A}) + \mathcal{B}_p$$

extended Bogolon states here do not carry current (cancelled by a superfluid backflow), reflecting the Meissner effect:

$\langle \mathbf{j} \rangle = 0 \Rightarrow$ if $\theta = 0$ (unitary gauge),

$$\frac{n_s}{m}\mathbf{A} = \langle \mathbf{j}^{\text{qp}} \rangle = v_F \widehat{\mathbf{k}}_0$$

Wave packet in the superconductor case

static wave packet $(\partial_t \rho^{\mathrm{qp}} = 0) \Rightarrow \frac{n_s}{m} \nabla \cdot \mathbf{A} = \langle \mathcal{B}_p \rangle = \langle \nabla \cdot \mathbf{j}^{\mathrm{qp}} \rangle$

Maxwell eq. $\nabla \times \mathbf{B} = 4\pi \mathbf{j} \quad \Rightarrow$

$$[-\nabla^2 + \lambda_{\mathrm{L}}^{-2}]\mathbf{A} = 4\pi \langle \mathbf{j}^{\mathrm{qp}} - \lambda_{\mathrm{L}}^2 \nabla(\nabla \cdot \mathbf{j}^{\mathrm{qp}}) \rangle$$

penetration depth: $\lambda_{\mathrm{L}}^{-2} = 4\pi \frac{n_s}{m}$

$$\langle \mathbf{j_q} \rangle_{\Psi} = v_F \left[\frac{q^2 \widehat{\mathbf{k}}_0 - (\mathbf{q} \cdot \widehat{\mathbf{k}}_0)\mathbf{q}}{q^2 + \lambda_{\mathrm{L}}^{-2}} \right] e^{-(\lambda^2 \mathbf{q}^2)/4}$$

exponential decay at long distances:

$$\langle \mathbf{j}(\mathbf{r}) \rangle_{\Psi} \propto e^{-r/\lambda_{\mathrm{L}}} \quad (r \gg \lambda, \lambda_{\mathrm{L}})$$

Paired Quantum Hall state

fermions in a (static) uniform background field $\mathbf{A}$ ($\mathbf{B} = \nabla \times \mathbf{A}$)
Chern-Simons term: $\mathcal{L}_{\rm CS} = (4\Phi_0)^{-1}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho$

density: $\rho = \rho^{\rm qp} - \kappa_0[\partial_t\theta - 2(a_0 + A_0)]$
current: $\mathbf{j} = \mathbf{j}^{\rm qp} + \frac{n_s}{2m}[\nabla\theta - 2(a + \mathbf{A})]$

total action $S + S_{\rm CS} \Rightarrow$ equations of motion: change $A_0 \to a_0 + A_0$, $\mathbf{A} \to \mathbf{a} + \mathbf{A}$ from the "superconductor" case

$$\partial_t\rho^{\rm qp} = -\nabla\cdot\mathbf{j}^{\rm qp} + \mathcal{B}_p$$

$$\kappa_0\partial_t[\partial_t\theta - 2(a_0 + A_0)] = \frac{n_s}{2m}\nabla\cdot[\nabla\theta - 2(\mathrm{a} + \mathrm{A})] + \mathcal{B}_p$$

$$[-\nabla^2 + \lambda_{\rm CS}^{-2}](\mathbf{a} + \mathbf{A}) = 8\kappa_0\Phi_0^2\langle\mathbf{j}^{\rm qp} - \lambda_{\rm CS}^2\nabla(\nabla\cdot\mathbf{j}^{\rm qp})\rangle$$

exponential decay with screening length: $\lambda_{\rm CS}^{-2} = 8\kappa_0\Phi_0^2\frac{n_s}{m}$

dipolar charge distribution stemming from $\mathbf{e} = -\partial_t a_0 - \nabla\mathbf{a} = 2\Phi_0\hat{z}\times\mathbf{j}$

Some remarks

- the results valid provided $\bar{\rho}^{\rm qp}\xi^2 \ll 1$
  (distance between quasiparticles is much larger than their size)

- Galilean-invariant system:
  $\mathbf{J}(\mathbf{x}) = \frac{e}{m}\mathbf{P}(\mathbf{x}) \Longrightarrow$ total current is ev_F
  [C. Nayak *et al.*, PRB **64**, 235113 (2001)]

- weak coupling theory in momentum space, vs. gauge-invariant strong-coupling case...