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Topological Entanglement Entropy with a Twist

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Defects in topologically ordered models have interesting properties that are reminiscent of the anyonic excitations of the models themselves. For example, dislocations in the toric code model are known as *twists* and possess properties that are analogous to Ising anyons. We strengthen this analogy by using the topological entanglement entropy as a diagnostic tool to identify properties of both defects and excitations in the toric code. Specifically, we show, through explicit calculation, that the toric code model including twists and dyon excitations has the same quantum dimensions, the same total quantum dimension, and the same fusion rules as an Ising anyon model.

Outline

- There are 2D spin lattice models for which
    - Excitations are anyonic quasiparticles
    - Energy eigenstates are topologically ordered

- Anyons are characterized by their quantum dimension

- The 'amount' of topological order (measured by the topological entanglement entropy) in a state is governed by the quantum dimensions of the relevant anyons

- It is possible to define defects in topological spin models. These are not excitations, but act a bit like anyons

- How do these effect the TEE?

Quantum Dimension

- An anyon model is defined by a set of particle types and fusion rules

$$\{1, a, b, c, \ldots\} \qquad a \times b = c + d$$

- The quantum dimensions of the anyons then satisfy

$$d_a \times d_b = d_c + d_d$$

- The total quantum dimension of the anyon model is defined as

$$D^2 = \sum_x d_x^2$$

- The dimension of the Hilbert space of N anyons of type *a* is then

$$\frac{d_a^N}{D^2}$$

Relevant Anyon Models

- Two anyon models are relevant to this work. First: *D(Z*$_2$*)*

$$\{1, e, m, \epsilon\} \qquad e\times e=m\times m=\epsilon\times\epsilon=1 \qquad e\times m=\epsilon$$

$$d_1=d_e=d_m=d_\epsilon=1 \qquad D=2$$

- Next: Ising

$$\{1, \sigma, \psi\} \qquad \psi\times\psi=1 \qquad \sigma\times\sigma=1+\psi \qquad \sigma\times\psi=\sigma$$

$$d_1=d_\psi=1 \qquad d_\sigma=\sqrt{2} \qquad D=2$$

Topological Entanglement Entropy

• Ground states of local, gapped 2D Hamiltonians satisfy the area law

• This states that, if we take the reduced density matrix of a region *A*, the von Neumann entropy will satisfy

$$S(\rho_A) = S_A = \alpha L - n\gamma \qquad S(\rho_A) = -\rho_A \log \rho_A$$

• Here *L* is the length of the boundary of *A*, and *n* is the number of disconnected boundary regions (typically 1)

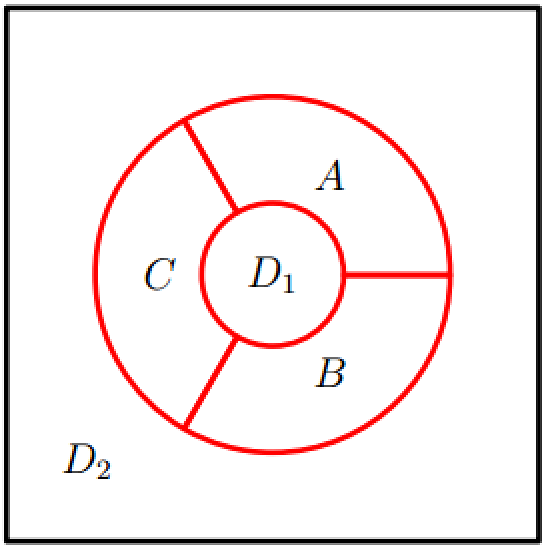
• The first term captures the short range correlations in the state. The second is non-zero only if there are *topological correlations* present

• If the excited states are consistent with an anyon model of dimension *D*

$$\gamma = \log D$$

Topological Entanglement Entropy

- By calculating the entropy for different regions, we can isolate the topological contribution



$$S_{ann} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{CA} + S_{ABC}$$
$$= -[I(A;BC) - I(A;B) - I(A,C)]$$
$$= -2\gamma = -2\log D$$

- This particular choice of regions has a nice feature: we can use it to analyze anyons in D_1

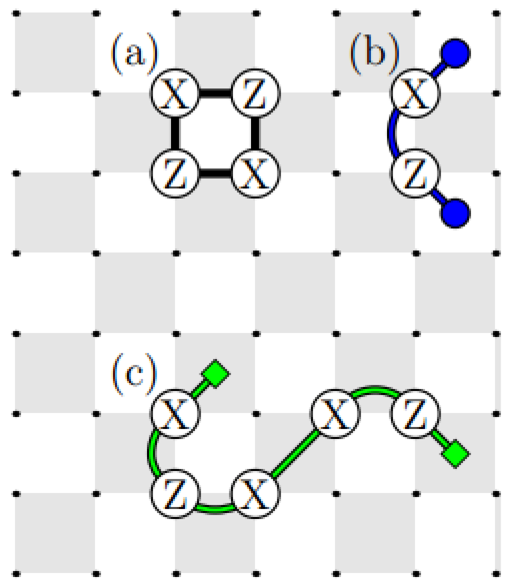
- When an anyon of type *a* is in there $S_{ann}(a) = -2\log D/d_a$

- When there are multiple anyons that fuse to a result x_1 with probability P_x, leaving x_2 in D_2

$$S_{ann}(\{x\}) = -2\log D - \sum_x P_x \log[P_x/(d_{x_1} d_{x_2})]$$

Spin Lattice Model

- We consider a spin lattice model whose excitations are $D(Z_2)$ anyons



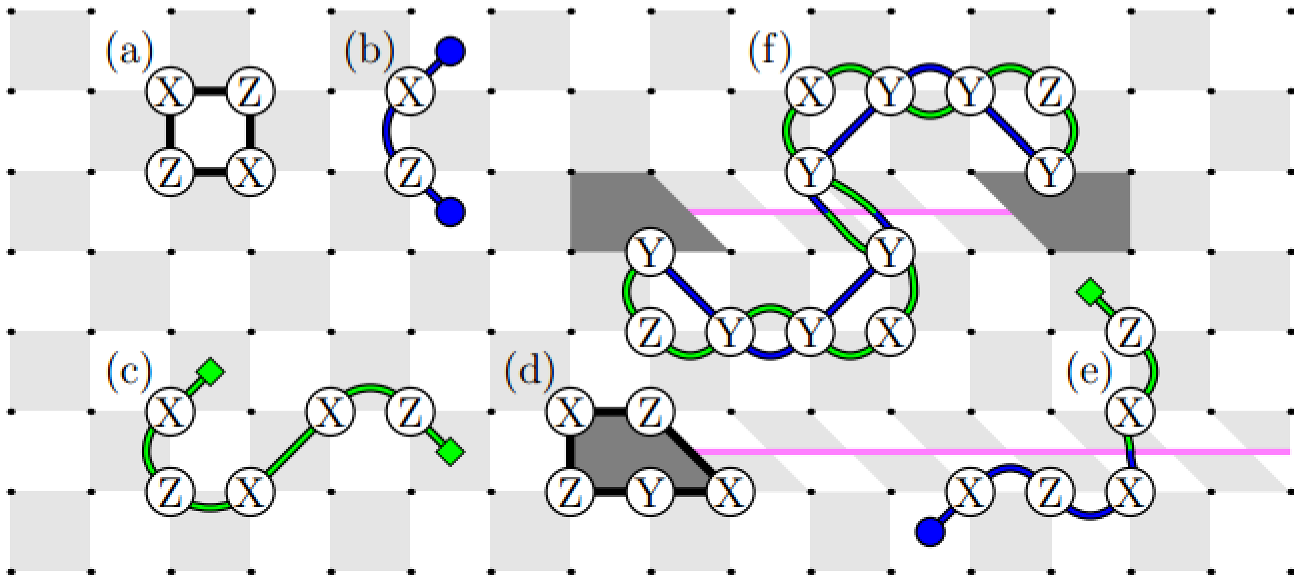
- (a) Stabilizer operators, used as terms in the Hamiltonian. Ground state is +1 eigenspace of all these. Excited states are plaquettes in -1 eigenspace.

- (b) Plaquettes are excited in pairs of the same colour. This is an example of an operator that creates two excited white plaquettes (electric charges, *e*)

- (c) Same for black plaquettes (magnetic charges, *m*). The ϵ particle is the composite of the two

Defects

- Adding the defects (d) allows charges to become fluxes, and vice-versa



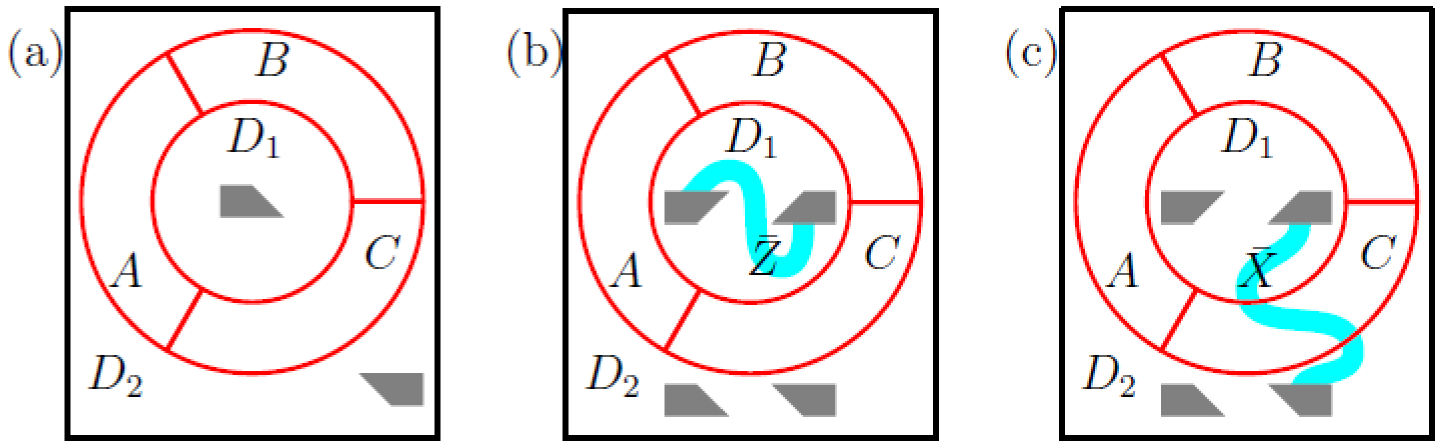
- This allows an ϵ to become 'hidden' in pairs of defects

- The defects (known as 'twists') therefore act like the σ particles of the Ising anyon model, and the ϵ behave like the ψ

- Does the TEE act accordingly?

Results

- The TEE was calculated for various configurations of twists



- (a) a single twist inside the annulus

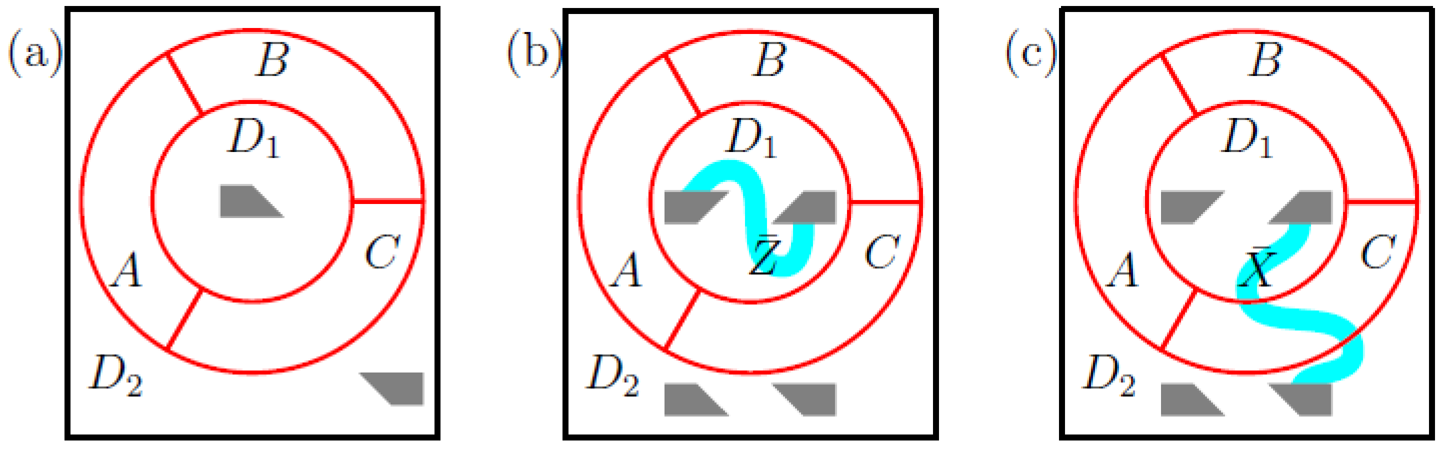
- Calculation yields $S_{ann}(\sigma) = -\log 2$

- This is exactly what would result if the twist was an Ising σ

$$S_{ann}(\sigma) = -2\log D/d_\sigma = -2\log 2/\sqrt{2} = -\log 2$$

Results

- The TEE was calculated for various configurations of twists



- (b) two connected twists, and hence collectively act like the vacuum

- Calculation yields

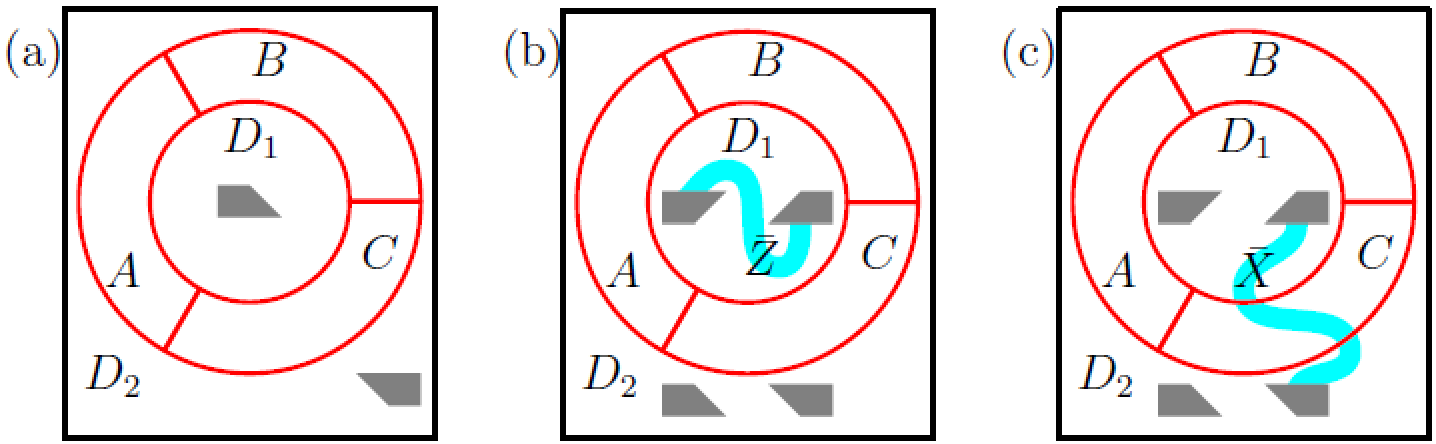
$$S_{ann}(\sigma\times\sigma=1)=-2\log 2$$

- This is exactly what would result if the two twists were collectively vacuum

$$S_{ann}(\sigma\times\sigma=1)=-2\log D/d_1=-2\log D$$

Results

- The TEE was calculated for various configurations of twists



- (b) two unconnected twists, and hence collectively act like the vacuum or a single fermion, both with probability *0.5*

- Calculation yields $S_{ann}(\sigma\times\sigma=1\vee\psi)=-\log 2$

- Again, exactly the same as the Ising case

$$S_{ann}(\sigma\times\sigma=1\vee\psi)=-2\log D-\sum_x P_x\log[P_x/(d_{x_1}d_{x_2})]$$

$$=-2\log 2-2\times 0.5\log[0.5/1]=-\log 2$$

Conclusions

- The TEE identifies properties of topologically ordered ground states
- It can also be used to identify properties of anyons (quantum dimension)
- Defects are considered with anyon like fusion behaviour
- This behaviour leads to an expected quantum dimension
- The behaviour of the TEE in the presence of the defects is exactly consistent with the expected quantum dimension