Sequential quantum-enhanced measurement with an atomic ensemble

A.V. Lebedev[1], P. Treutlein[2], and G. Blatter[1]
[1]ETH Zürich, [2]Uni Basel
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Journal Club

avuba

- Email last Wednesday
- Assistants' association of the University of Basel
  - Represent interests with the authorities & the public.
    - No further details so far
  - Automatic membership
  - Cost: 15 CHF each semester (added to the semester fee)
- Option: decline membership by sending a letter (with matriculation number)
  - Deadline: 30^{th} of April

UNIVERSITÄT BASEL

DER REKTOR

To all doctoral students and assistants

Prof. Dr. Antonio Loprieno

Petersgraben 35
Postfach
4003 Basel
Switzerland

Tel. +41 (0)61 267 30 30
Fax +41(0)61 267 30 03
E-Mail: A.Loprieno@unibas.ch
http://www.unibas.ch

Basel, April 2013

Your membership in *avuba*, the assistants' association of the University of Basel

Dear doctoral students and assistants,

With your re-enrollment in a doctoral program at the University of Basel, in Fall 2013 you will automatically become a member of the newly-established assistants' association of the University of Basel (*avuba*, Assistierendenvereinigung der Universität Basel). According to the University Statute (§ 22), the mission of *avuba* is to represent the interests of assistants and doctoral students with the authorities and the public. The association also provides a variety of services and participates in the election of representatives to the standing committees of the University.

As a member of *avuba*, you will pay a membership contribution of CHF 15 each semester. This amount will be added to the semester fees unless you waive your membership by writing to the Rector's Office with your matriculation number included in the correspondence (Address: Universität Basel, Rektorat, avuba, Petersgraben 35/4, 4003 Basel). Specific deadlines apply. If you wish to waive your membership for the Fall 2013 semester, you must do so by the 30^{th} of April. Please note, however, that in so doing you renounce your right be represented by *avuba*.

The participation of doctoral students and assistants is of great importance at all levels, including departments, faculties, and University committees. We therefore encourage you to make use of your rights and to support the work of *avuba* with your membership and your willingness to fulfill tasks within the association and the self-administration of the University. *avuba* looks forward to answering any questions you might have. Please find the contact details in the course directory or at http://avuba.unibas.ch.

With best regards,

Prof. Dr. Antonio Loprieno
Rector

UNI BASEL

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Introduction

- Quantum channel $$\hat{\rho}_0 \rightarrow \mathcal{Q}_\phi(\hat{\rho}_0) = \hat{\rho}_\phi$$
- Repeat measurement $$\hat{\rho}_0^{\otimes R} \rightarrow \left[\mathcal{Q}_\phi(\hat{\rho}_0)\right]^{\otimes R} = \hat{\rho}_\phi^{\otimes R}$$
  - Increase precision
    - Standard quantum limit / shot noise limit
      $$\delta_0 \rightarrow \delta_0/\sqrt{R}$$
    - Nature provides Heisenberg limit
      $$\delta_0 \rightarrow \delta_0/R$$
      exploit quantum resources

Quantum enhancement in the measurement precision

- Parallel strategy:
  - Division into m sub-ensembles with n probes
  - $[\hat{\rho}_0^{(n)}]^{\otimes m} \rightarrow [\mathcal{Q}_\phi(\hat{\rho}_0^{(n)})]^{\otimes m}$
  - Exploit entanglement
  - Drawback: fragile entangled states

- Sequential strategy:
  - Each of m separate probes is passed n times through the same quantum channel.
  - $\hat{\rho}_0^{\otimes m} \rightarrow [\mathcal{Q}_\phi^n(\hat{\rho}_0)]^{\otimes m}$
  - Exploit coherent quantum dynamics
  - Drawback: longer coherence times required

$\rightarrow$ same enhancement in precision.

Approach

- Sequential strategy without entanglement
  - Reach given precision δ in K steps.

$$K \sim \ln(\delta_1/\delta)/\ln(\sqrt{N}) \qquad \delta_1 = \delta_0/\sqrt{N}$$

- resembles Kitaev's phase estimation algorithm
- Less steps due to large ensemble of individual probes in the atom cloud
- Relies on collective manipulations and measurements

Primary measurement

- Unknown unitary rotation $\hat{U}_z[\phi] = \exp[-i\hat{\sigma}_z\phi/2]$
- Ramsey interferometry as primary measurement
- Prepare the polarized state $\hat{\sigma}_x = 1$ $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$
- Applying the rotation $[(e^{-i\phi/2}|\uparrow\rangle + e^{+i\phi/2}|\downarrow\rangle)/\sqrt{2}]^{\otimes N}$
- Read-out state $\hat{U}_y[-\pi/2]$ $[\cos(\phi/2)\,|\uparrow\rangle + i\sin(\phi/2)\,|\downarrow\rangle]^{\otimes N}$

Primary measurement: total polarization

- $\hat{S}_z = (1/N) \sum_{i=1}^{N} \hat{\sigma}_z^{(i)}$ $\qquad S_z = [N_+ - N_-]/N$
- The probability to observe a particular value is given by the Bernoulli distribution

$$\mathcal{P}(S_z = \tilde{S}_z|\phi) = \frac{N!}{\tilde{N}_+!\tilde{N}_-!}\left(\cos^2\frac{\phi}{2}\right)^{\tilde{N}_+}\left(\sin^2\frac{\phi}{2}\right)^{\tilde{N}_-}$$

- Variable change & use of Bayes' theorem $\phi \to p = \cos\phi$

$$P(p|\tilde{S}_z) = \frac{(N+1)!}{2\tilde{N}_+!\tilde{N}_-!}\left(\frac{1+p}{2}\right)^{\tilde{N}_+}\left(\frac{1-p}{2}\right)^{\tilde{N}_-}$$

sharp peak for large $\tilde{N}_+,\ \tilde{N}_-$

Primary measurement: expansion

- Expansion around the maximum and replacement with normal distribution

$$P(p|\tilde{S}_z) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(p-\tilde{S}_z)^2}{2\sigma^2}\right], \quad \sigma^2 = \frac{1-\tilde{S}_z^2}{N}, \quad (3)$$

  - With tolerance level $1-\beta = \mathrm{erf}(g/\sqrt{2})$ the precision of the estimate $p \approx \tilde{S}_z$ is given by $\mathrm{Prob}\left[|p-\tilde{S}_z| \leq g(\beta)\sigma\right] = 1-\beta,$

- (3) provides the distribution function for the angle

$$P(\phi|\tilde{S}_z) = \frac{|\sin\phi|}{2\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(\cos\phi-\tilde{S}_z)^2}{2\sigma^2}\right]. \qquad (5)$$

Complementary measurement

- Two symmetric intervals

$$|\phi \pm \tilde{\phi}| \leq g/\sqrt{N} \text{ with } \tilde{\phi} = |\arccos(\tilde{S}_z)|$$

- Expanding (5), the distribution of the angle Φ is given by the sum of two normal distributions

$$\phi \sim \frac{1}{2} \sum_{\alpha=\pm 1} \mathcal{N}(\alpha\tilde{\phi}, \sigma_1^2), \qquad \sigma_1 = \frac{1}{\sqrt{N}},$$

describing two equiprobable alternatives for the angle.

Complementary measurement: second test

- Prepare the ensemble $\sigma_y = +1$ $\quad ([|\uparrow\rangle + i|\downarrow\rangle]/\sqrt{2})^{\otimes N'}$
- Repeat Ramsey measurement
  - Parameter $p' = \sin\phi$
- Given the probability

$$\mathcal{P}(\hat{\sigma}_z = \pm 1|\alpha\tilde{\phi}) = \frac{1}{2}\int d\phi \, \frac{1 \pm \sin\phi}{2} \sum_{\alpha=\pm 1} \mathcal{N}(\alpha\tilde{\phi}, \sigma_1^2)$$
$$\approx \left[1 \pm \left(\alpha \sin\tilde{\phi} - \sin^2\tilde{\phi}\,/2N\right)\right]/2, \quad (7)$$

the total polarization is given by $S'_z \sim \sum_\alpha \mathcal{N}(S'_{z\alpha}, \sigma'^2)/2,$

with mean and variance $S'_{z\alpha} \approx \alpha \sin\tilde{\phi} - \dfrac{\sin^2\tilde{\phi}}{2N}, \quad \sigma'^2 \approx \dfrac{\cos^2\tilde{\phi}}{N'}.$

Complementary measurement classification rule

- Define regions

$$E_- = \{S'_z | S'_z < \bar{S}'_z\}$$
$$E_+ = \{S'_z | S'_z > \bar{S}'_z\}$$

$$\mathcal{P}(E_-) \equiv \int_{-\infty}^{\bar{S}'_z} dS'_z \, P(S'_z) = \mathcal{P}(E_+) = 1/2$$

- Misclassification error $\beta' = \mathcal{P}(+|E_-)\mathcal{P}(E_-) + \mathcal{P}(-|E_+)\mathcal{P}(E_+)$

$$\beta' = [1 - \mathrm{erf}(|\sin\tilde{\phi}|/\sqrt{2}\sigma')]/2$$

- Away from the immediate vicinity of $\tilde{\phi} \approx 0$ (e.g., $\tilde{N}_+ > 5$) and a typical number of probes N ~ 10^3, choosing N' ~ N results in a negligible probability of misclassification.

n-fold rotation

- Sequential application of the rotation $\hat{U}_z[\phi]$
- Ensemble polarization gives estimate for $\phi_n = n\phi$
- Given a measured polarization, the distribution function is again given by (5)

$$P(\phi|\tilde{S}_z) = \frac{|\sin\phi|}{2\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(\cos\phi - \tilde{S}_z)^2}{2\sigma^2}\right]. \qquad (5)$$

$$\tilde{S}_z \rightarrow \tilde{S}_{zn}$$
$$\phi \rightarrow \phi_n$$

  providing two Gaussian peaks
- The complementary measurements again selects one alternative.

n-fold rotation a further estimate

- Reading (5) with $\cos\phi \to \cos(n\phi)$

$$P(\phi|\tilde{S}_z) = \frac{|\sin\phi|}{2\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(\cos\phi - \tilde{S}_z)^2}{2\sigma^2}\right]. \qquad (5)$$

  n different values of Φ to the same value of cos(nΦ)

- The distribution function for the angle Φ then has n peaks

$$\phi \sim \frac{1}{n}\sum_{k=0}^{n-1} \mathcal{N}(\tilde{\phi}_{nk}, \sigma_n^2), \quad \sigma_n = \frac{\sigma_1}{n}, \quad \tilde{\phi}_{nk} = \frac{\tilde{\phi}_n}{n} + 2\pi\frac{k}{n}$$

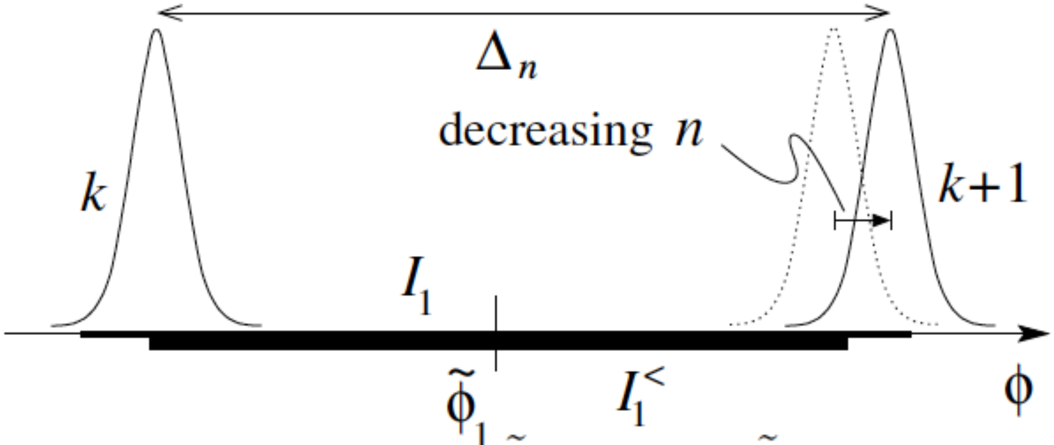
  The n-fold measurement redistributes the original uncertainty $\delta_1 \sim \sigma_1$ among n different alternatives.

- The different distribution of the probability allows gain in precision when combining the two measurements.

Beyond the shot noise limit

- Identify correct alternative of the n-fold measurement
  → combine 1- and n-fold measurements.
- Define interval centered around the result of the first measurement
  $$I_1 = \{\phi| \; |\phi - \tilde{\phi}_1| \leq g(\beta)\sigma_1\}$$
  $$\mathrm{Prob}[|\phi - \tilde{\phi}_1| \; \leq g(\beta)\sigma_1] = 1 - \beta$$
- The alternative A_k is compatible with the first measurement for $\mathrm{Prob}\left[\phi \in I_1 | \phi \sim \mathcal{N}\left(\tilde{\phi}_{nk}, \sigma_n^2\right)\right] \geq 1 - \beta$
- Satisfy the original confidence in the 2[nd] measurement
  → reduced interval $I_1^< = \{\phi| \; |\phi - \tilde{\phi}_1| \leq g(\beta)(\sigma_1 - \sigma_n)\}$

Beyond the shot noise limit



$$\tilde{\phi}_{nk} = \tilde{\phi}_1 - g(\beta)(\sigma_1 - \sigma_n) \qquad \tilde{\phi}_{nk+1} = \tilde{\phi}_1 - g(\beta)(\sigma_1 - \sigma_n) + \Delta_n$$

- Optimum: $\tilde{\phi}_{nk+1} = \tilde{\phi}_1 + g(\beta)(\sigma_1 - \sigma_n)$
- Find largest n compatible with $\tilde{\beta} \ll 1$

$$n \leq n_{\text{opt}} = \lfloor \nu(\beta, \tilde{\beta})/\sigma_1 \rfloor$$

$$\nu = \pi g(\beta) \frac{\sqrt{1 + 2\ln[(1-\tilde{\beta})/\tilde{\beta}]/g^2(\beta)} - 1}{\ln[(1-\tilde{\beta})/\tilde{\beta}]}$$

Beyond the shot noise limit - n_{opt}

- 1- and n-fold combined

$$P(\phi|\tilde{\phi}_1, \tilde{\phi}_n) \propto \sum_{k=0}^{n-1} \frac{w_k}{\sqrt{2\pi}\,\sigma_{1,n}} \exp\left[-\frac{(\phi - \tilde{\phi}_{1,nk})^2}{2\sigma_{1,n}^2}\right]$$

with

$$\tilde{\phi}_{1,nk} = (\tilde{\phi}_1\sigma_n^2 + \tilde{\phi}_{nk}\sigma_1^2)/(\sigma_1^2 + \sigma_n^2) \approx \tilde{\phi}_{nk}$$

$$\sigma_{1,n} = \sigma_1\sigma_n/\sqrt{\sigma_1^2 + \sigma_n^2} \approx \sigma_n$$

$$w_k = \exp\left[-\frac{(\tilde{\phi}_1 - \tilde{\phi}_{nk})^2}{2(\sigma_1^2 + \sigma_n^2)}\right]$$

- The misclassification error is given by

$$\tilde{\beta} = w_{k+1}/(w_k + w_{k+1})$$

→ solve for n_{opt}

Estimates & Comparison

- Precision estimates:
  - 1-fold & n_{opt}fold $\quad \mathrm{Prob}\big[|\phi - \tilde{\phi}_2| \le g(\beta)\sigma_2\big] = (1-\beta)(1-\tilde{\beta}),$
  - K iterations $\quad \mathrm{Prob}\big[|\phi - \tilde{\phi}_K| \le g(\beta)\sigma_K\big] = (1-\beta)(1-\tilde{\beta})^{K-1},$
- Kitaev's phase estimation algorithm
  - # steps: $K \sim \ln(1/\delta)/\ln 2$
  - Resources: $\delta \sim \ln R/R$
- Ensemble-based protocol
  - # steps: $K \sim \ln(1/\delta)/\ln\sqrt{N}$
  - Resources: $\delta \sim R^{-K/(K+1)}$
  - Few iterations on an ensemble of N ~ 10^3 atoms with long coherence times.

Conclusion

- The sequential strategy discussed here is particularly useful in scanning probe measurements of a spatially varying field.
- The scheme dramatically reduces the overall time to record a picture with a given precision.
- The ensemble-based sequential strategy could also be combined with a parallel strategy using squeezed BECs.