PRL **110,** 010502 (2013)

Surface Code Threshold in the Presence of Correlated Errors

E. Novais[1] and Eduardo R. Mucciolo[2]
[1]*Centro de Ciências Naturais e Humanas, Universidade Federal do ABC, Santo André, São Paulo 09210-170, Brazil*
[2]*Department of Physics, University of Central Florida, Orlando, Florida 32816, USA*
(Received 6 September 2012; published 2 January 2013)

We study the fidelity of the surface code in the presence of correlated errors induced by the coupling of physical qubits to a bosonic environment. By mapping the time evolution of the system after one quantum error correction cycle onto a statistical spin model, we show that the existence of an error threshold is related to the appearance of an order-disorder phase transition in the statistical model in the thermodynamic limit. This allows us to relate the error threshold to bath parameters and to the spatial range of the correlated errors.

Fidelity of the surface code in the presence of a bosonic bath

P. Jouzdani,[1] E. Novais,[2] and E. R. Mucciolo[1]
[1]*Department of Physics, University of Central Florida, Orlando, Florida 32816-2385, USA*
[2]*Centro de Ciências Naturais e Humanas, Universidade Federal do ABC, Santo André, SP, Brazil*
(Dated: April 11, 2013)

We study the resilience of the surface code to decoherence caused by the presence of a bosonic bath. This approach allows us to go beyond the standard stochastic error model commonly used to quantify decoherence and error theshold probabilities in this system. The full quantum mechanical system-bath dynamics is computed exactly over one quantum error correction cycle. Since all physical qubits interact with the bath, space-time correlations between errors are taken into account. We compute the fidelity of the surface code as a function of the quantum error correction time. The calculation allows us to map the problem onto an Ising-like statistical spin model with two-body interactions and a fictitious temperature which is related to the inverse bath coupling constant. The model departs from the usual Ising model in the sense that interactions can be long ranged and can involve complex exchange couplings; in addition, the number of allowed configurations is restricted by the syndrome extraction. Using analytical estimates and numerical calculations, we argue that, in the limit of an infinite number of physical qubits, the spin model sustain a phase transition which can be associated to the existence of an error threshold in the surface code. An estimate of the transition point is given for the case of nearest-neighbor interactions.

Motivation

- The surface code is the most promising quantum error correction (QEC) code:
  - Requires ‘only’ nearest-neighbor two-qubit entangling gates and single-qubit control
  - High error threshold
- Often studied with simplistic, stochastic error models
- E.g. assuming an i.i.d. error model of the form

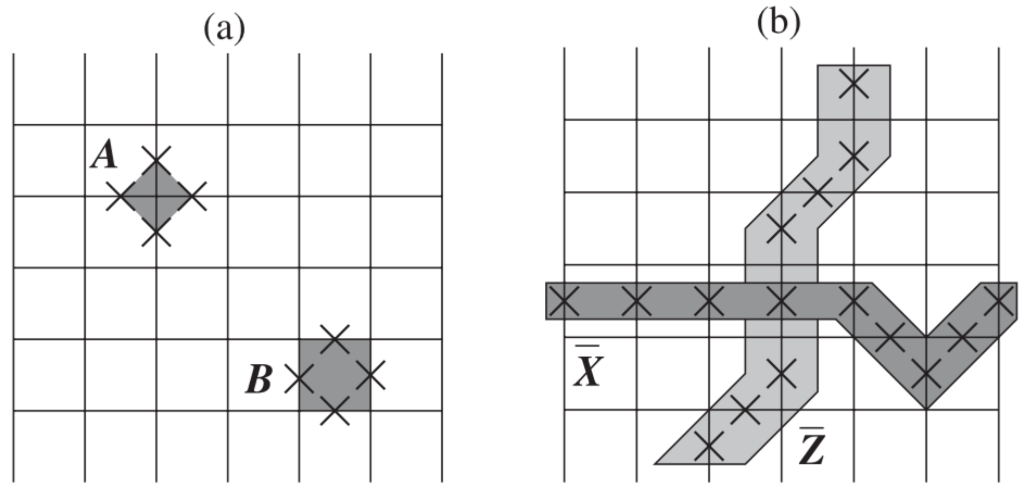
$$\rho \mapsto p_I\rho + p_x\sigma_x\rho\sigma_x + p_y\sigma_y\rho\sigma_y + p_z\sigma_z\rho\sigma_z$$

allows to derive error thresholds and to benchmark error correction algorithms.

Goal

- Test the resilience of the surface code against an error model which arises from a microscopic description of the environment.
- Caldeira-Leggett type of environment, **freely propagating bosonic modes** (-> physical motivation: photons, phonons)
- Study the fidelity of the surface code after one QEC cycle as a function of the duration of each QEC cycle.
- Simplifying assumptions: The bath is at **zero temperature** and is reset to its groundstate after each QEC cycle.

Surface code reminder



- A and B: stabilizer operators: required to yield $+1$ eigenvalues; measured after each QEC cycle

$$A_\Diamond = \prod_{i \in \Diamond} \sigma_i^x \qquad B_\Box = \prod_{i \in \Box} \sigma_i^z$$

- $\bar{X}$ and $\bar{Z}$: examples of logical operators

Model

$$H = H_0 + V \qquad H_0 = \sum_{\mathbf{k}\neq 0} \omega_{\mathbf{k}}\, a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}$$

$$V = \frac{\lambda}{2} \sum_i f\left(\mathbf{r}_i\right) \sigma_i^x$$

Coupling between bosonic modes and code qubits

$$f\left(\mathbf{r}\right) = \frac{\left(v/\omega_0\right)^{D/2+s}}{L^{D/2}} \sum_{\mathbf{k}\neq 0} \left|\mathbf{k}\right|^s \left(e^{i\mathbf{k}\cdot\mathbf{r}} a^{\dagger}_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{r}} a_{\mathbf{k}}\right)$$

- D: bath spatial dimension, L: linear size of the code
- $\omega_{\mathbf{k}} = v|\mathbf{k}|$
- $s = -\frac{1}{2}$: coupling to the bosonic displacement field
- $s = +\frac{1}{2}$: coupling to the bosonic current operator
- $s = 0$: local creation and destruction of bosons

Dynamics

- Δ: duration of one QEC period
- Evolution operator in the interaction picture:

$$U(\Delta) = T_t \, \exp\left[-i\frac{\lambda}{2}\int_0^{\Delta} dt \sum_i f\left(\mathbf{r}_i, t\right)\sigma_i^x\right]$$

- In order to get rid of the time-ordering operator, perform a **Magnus expansion**, where due to the bosonic nature of the f-operators only the first and second term are non-vanishing.

Dynamics, ctd.

Result for the evolution operator in the interaction picture:

$$U(\Delta) = \chi \exp\left[-\frac{\lambda^2}{2}\sum_{i\neq j}\Phi_{\mathbf{r}_i\mathbf{r}_\mathbf{j}}(\Delta)\,\sigma_i^x\sigma_j^x\right] : \exp\left[-\frac{i\lambda}{2}\sum_i F_{\mathbf{r}_i}(\Delta)\,\sigma_i^x\right] :$$

bath correlation functions

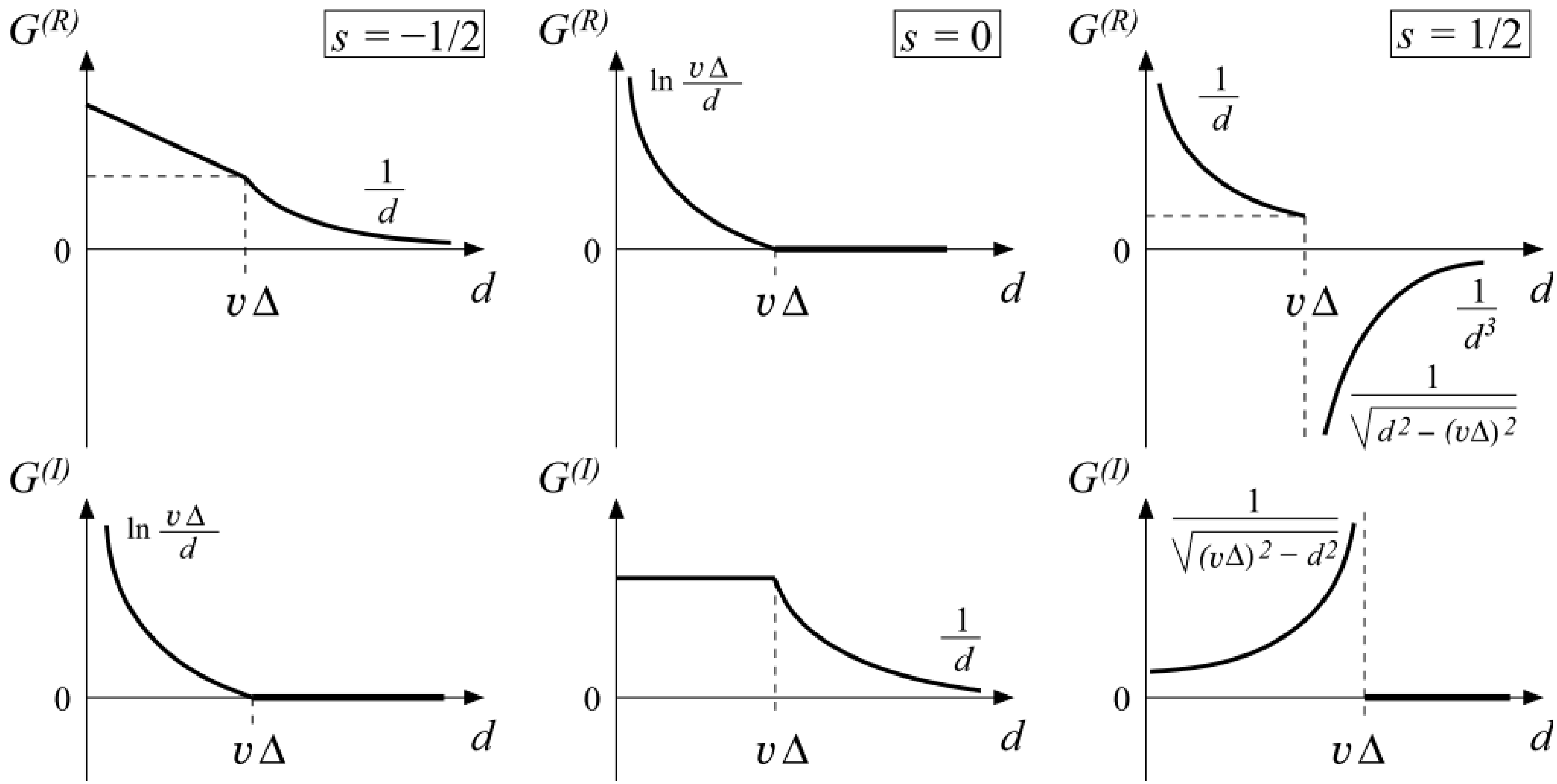
$$\Phi_{\mathbf{rs}}(\Delta) = \frac{1}{2}\left[\mathcal{G}^{(R)}_{\mathbf{rs}}(\Delta) + \mathcal{G}^{(I)}_{\mathbf{rs}}(\Delta)\right],$$

$$F_{\mathbf{r}}(\Delta) = \int_0^\Delta dt\, f\left(\mathbf{r}, t\right)$$

$$\chi = \exp\left[-\frac{\lambda^2}{4}\sum_i \Phi_{\mathbf{r}_i\mathbf{r}_i}(\Delta)\right],$$

Correlation functions for D=2

subohmic **ohmic** **superohmic**



- With the bosons initially at finite temperature, all correlation functions decay over a length-scale v/T.

Syndrome extraction

- After a time Δ, all stabilizer operators are measured (‘the syndrome is extracted’) flawlessly
- (Strong) simplifying assumption: all stabilizer operators yield a non-error syndrome -> Either the memory is still in its initial state or a logical operator $\bar{X}$ has been applied.
- Syndrome extraction is equivalent to the application of the operator

$$P' = |\Psi_0\rangle\langle\Psi_0| + \bar{X}|\Psi_0\rangle\langle\Psi_0|\bar{X}$$

Fidelity

- The fidelity of the surface after one QEC cycle is $F \equiv |\langle \Psi_{\mathrm{QEC}} | \Psi_0 \rangle|$

$$|\Psi_{\mathrm{QEC}}\rangle = P' U(\Delta) |\Psi_0\rangle$$

$$F = \frac{|\mathcal{A}|}{\sqrt{|\mathcal{A}|^2 + |\mathcal{B}|^2}}$$

$$\mathcal{A} = \langle \Psi_0 | U(\Delta) | \Psi_0 \rangle$$

$$\mathcal{B} = \langle \Psi_0 | \bar{X} U(\Delta) | \Psi_0 \rangle$$

Protected code space

$$|\bar{\uparrow}\rangle = G|F_z\rangle \qquad |\bar{\downarrow}\rangle = G\bar{X}|F_z\rangle$$

'codewords', basis states of the code subspace

$$G = \frac{1}{\sqrt{2^{N_\Diamond}}}\prod_{\Diamond}(1 + A_\Diamond) \qquad |F_z\rangle = \prod_{i=1}^{N} |\uparrow\rangle_{i,z}$$

$$\prod_{\Diamond}(1 + A_\Diamond) = 1 + \sum_{\Diamond} A_\Diamond + \sum_{\Diamond_1 \neq \Diamond_2} A_{\Diamond_1} A_{\Diamond_2} + \ldots .$$

Mapping onto a statistical model

$$\mathcal{A} = \chi \left\langle F_z \left| e^{-\beta \mathcal{H}} \, G^2 \right| F_z \right\rangle \qquad \mathcal{B} = \chi \left\langle F_z \left| \bar{X} e^{-\beta \mathcal{H}} \, G^2 \right| F_z \right\rangle$$

$$|F_z\rangle = \prod_{i=1}^{N} \left(\frac{|\uparrow\rangle_{i,x} + |\downarrow\rangle_{i,x}}{\sqrt{2}} \right)$$

Example: Ohmic bath ($s = 0$):

$$\beta = \frac{1}{2\pi} \left(\frac{\lambda}{\omega_0} \right)^2$$

'fictitious' inverse temperature

$$\mathcal{H} = \sum_{i \neq j} J_{ij} \, \sigma_i^x \sigma_j^x \qquad J_{ij} = \frac{1}{2} \times \begin{cases} \operatorname{arcosh}\left(\frac{v\Delta}{|\mathbf{r}_i - \mathbf{r}_j|} \right) + \frac{i\pi}{2}, & \frac{|\mathbf{r}_i - \mathbf{r}_j|}{v\Delta} < 1, \\ i \arcsin\left(\frac{v\Delta}{|\mathbf{r}_i - \mathbf{r}_j|} \right), & \frac{|\mathbf{r}_i - \mathbf{r}_j|}{v\Delta} > 1. \end{cases}$$

→ Need to evaluate expectation values of a statistical spin model with complex two-body interactions and a restricted (due to G^2) configuration space

Phase transition

- It is expected that in the termodynamic limit there is a critical β_c such that

$$F = \begin{cases} 1, & \beta < \beta_c \\ 1/\sqrt{2}, & \beta > \beta_c \end{cases}$$

- In order to find β_c, study a simplified model

$$J_{\mathbf{rs}} = \begin{cases} J, & \mathbf{r}, \mathbf{s} \text{ nearest neighbors,} \\ 0 & \text{otherwise,} \end{cases}$$

where J is real.

- This is approriate if $v\Delta \approx a$ and imaginary parts are irrelevant.

Phase transition - Results

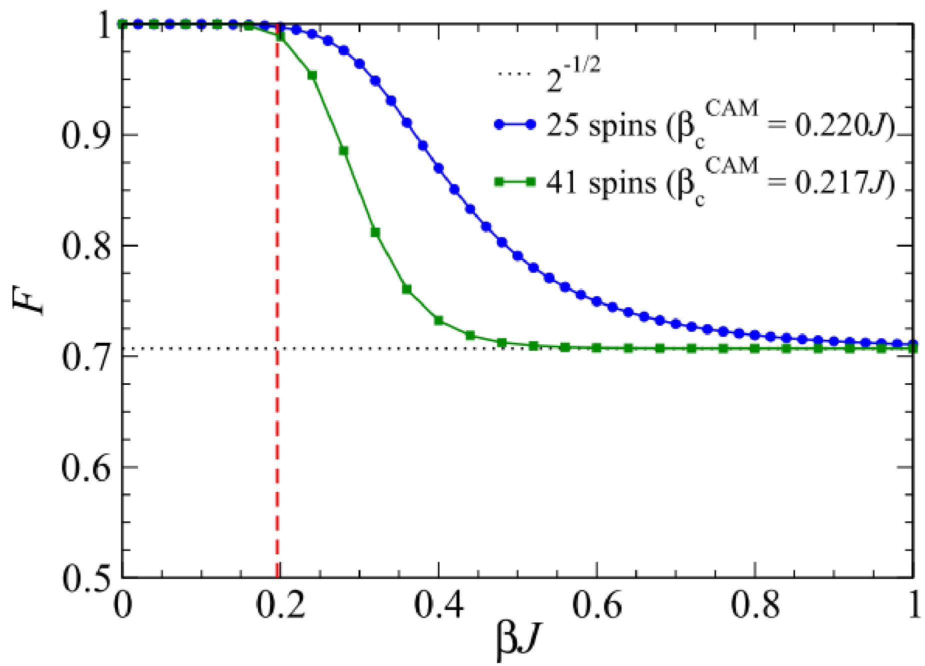


FIG. 3. (Color online) Surface code fidelity of code spaces of 25 and 41 physical qubits in contact with a bosonic bath when star operators are restricted to positive values ($A_\Diamond = 1$).

Result: $\beta_c J \approx 0.2$

The vertical red line corresponds to a mean field solution (Coherent Anomaly Method)

Influence of imaginary part

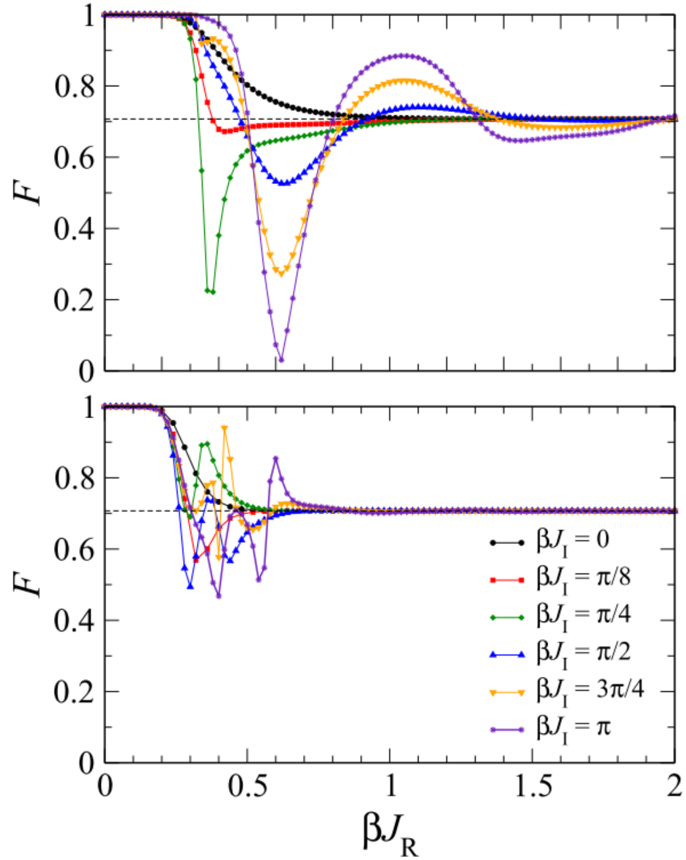


FIG. 4. (Color online) Fidelity of a code space of 25 physical qubits in contact with a bosonic bath when star operators are restricted to positive values ($A_\Diamond = 1$) and an imaginary part is added to the coupling constant: $J = J_R + iJ_I$. The data sets correspond to different values of J_I.

Conclusion: Adding a constant imaginary part to J_{rs} leads to oscillations close to the critial region that become smaller with increasing system size.

Conclusion

- Under the assumptions of
  - resetting the bath to its groundstate after each QEC cycle, and
  - a trivial error syndrome being measured

  there is a non-trivial mapping from the fidelity of the surface code to the evaluation of expectation values for a statistical spin model with complex two-body interactions and a restricted configuration space.

Conclusion

- This spin model is argued to undergo a disorder (fidelity 1) to order (fidelity $1/\sqrt{2}$) transition, depending on
  - Δ (duration of one QEC cycle)
  - v (mode velocity)
  - a (code lattice constant)
  - λ (coupling strength)
  - ω_0 (characteristic bath frequency)

  (Δ, v, a →) number of qubits within effective light cone → effective qubit interaction J_{ij}

- Fictitious inverse temperature

$$\beta = \frac{1}{2\pi}\left(\frac{\lambda}{\omega_0}\right)^2 \frac{1}{(\omega_0\Delta)^{D+2s-2}}$$

- For $J_{ij} = \mathrm{J}$ (nearest neighbors) the transition is found to happen at $\beta_c J \approx 0.2$