# Steady-State Entanglement in the Nuclear Spin Dynamics of a Double Quantum Dot

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(Dated: August 15, 2013)

We propose a scheme for the deterministic generation of steady-state entanglement between the two nuclear spin ensembles in an electrically defined double quantum dot. Due to quantum interference in the collective coupling to the electronic degrees of freedom, the nuclear system is actively driven into a two-mode squeezed-like target state. The entanglement build-up is accompanied by a self-polarization of the nuclear spins towards large Overhauser field gradients. Moreover, the feedback between the electronic and nuclear dynamics leads to multi-stability and criticality in the steady-state solutions.

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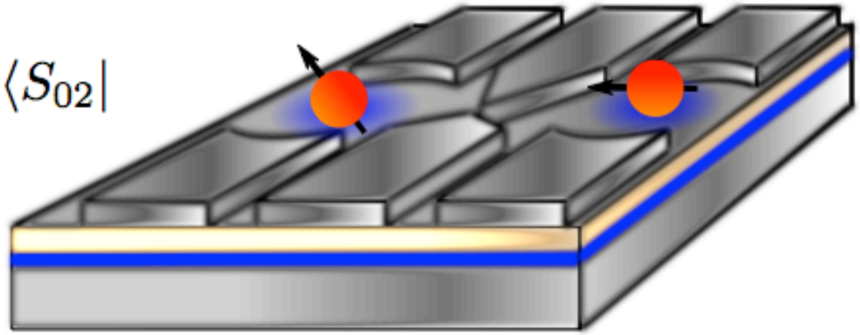
# System

- Double Quantum Dot

$$H_{\rm el} = \omega_0 \left(S_1^z + S_2^z\right) + \Delta \left(S_2^z - S_1^z\right) - \epsilon \left|S_{02}\right\rangle \left\langle S_{02}\right| + t\left(\left|\Uparrow\Downarrow\right\rangle \left\langle S_{02}\right| - \left|\Downarrow\Uparrow\right\rangle \left\langle S_{02}\right| + \mathrm{h.c.}\right),$$



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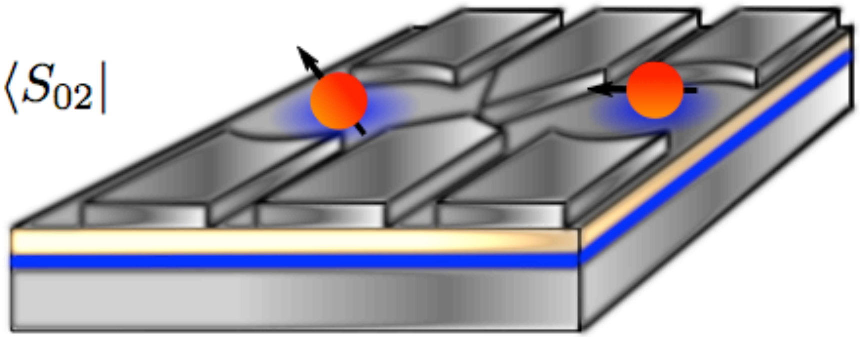
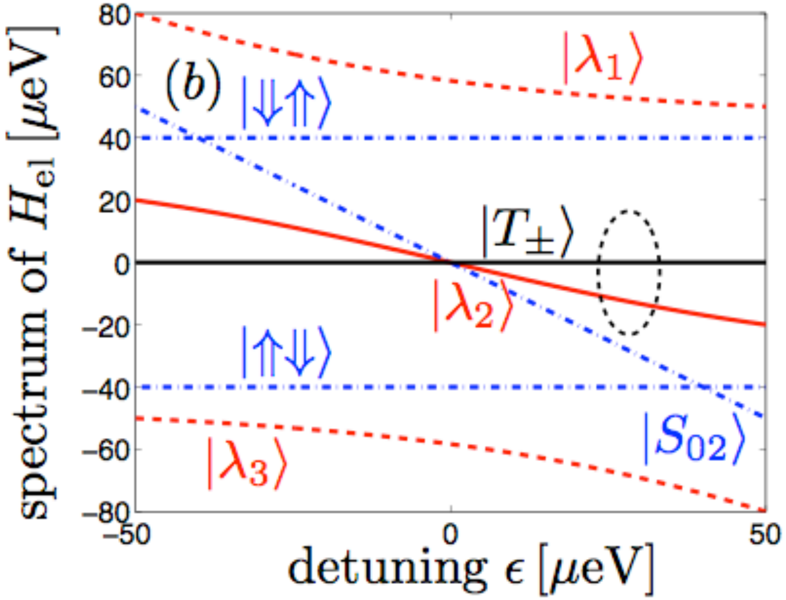


- Spectrum of $H_{el}$




Eigenstates in $S^z_{\rm tot} = 0$ are in red

# Generating entanglement

- Electron spins are not isolated (interact with nuclear spins)

$$H_{\mathrm{HF}} = \frac{a_{\mathrm{hf}}}{2} \sum_{i=1,2} \left(S_i^+ A_i^- + S_i^- A_i^+\right) + a_{\mathrm{hf}} \sum_{i=1,2} S_i^z A_i^z$$

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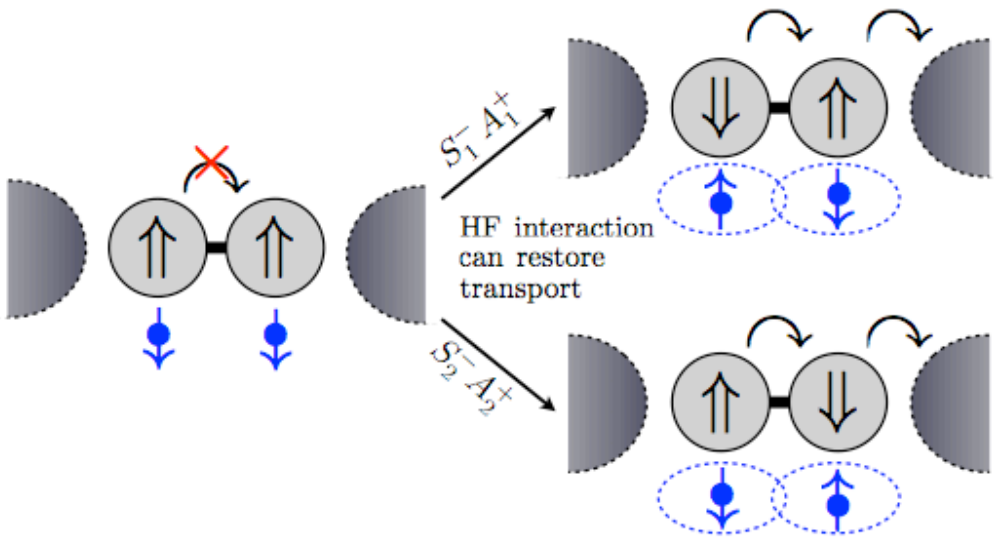
- Effect of Hyperfine Interaction in Transport Setup



# Generating entanglement (2)

- How do we get rid of the electronic states? Dissipation

Plenio, Huelga, Beige, and Knight, PRA **59**, 2468 (1999)

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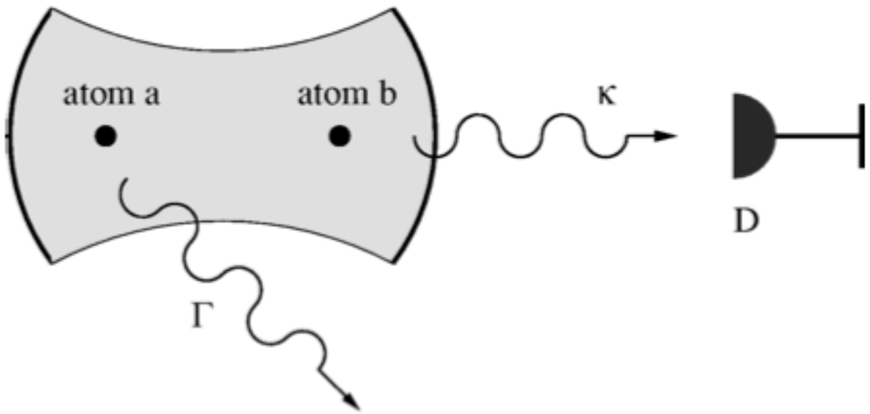
Plenio, Huelga, Beige, and Knight, PRA **59**, 2468 (1999)

- Basic idea




$$|\lambda_0\rangle = \frac{1}{\sqrt{g_a^2 + g_b^2}} \left( g_a |001\rangle - g_b |010\rangle \right),$$

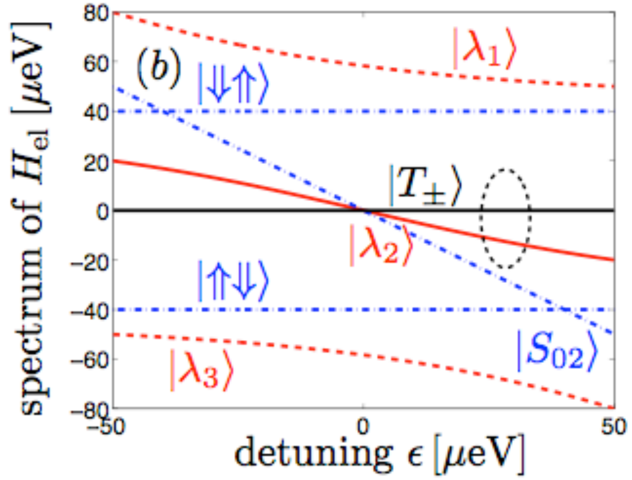
$$|\lambda_{1,2}\rangle = \frac{1}{\sqrt{2}} \left( |100\rangle \pm \frac{i}{\sqrt{g_a^2 + g_b^2}} \left( g_b |001\rangle + g_a |010\rangle \right) \right).$$

# Nuclear State Dynamics

- Effective flip-flop Hamiltonian

$$H_{\mathrm{ff}} = \frac{a_{\mathrm{hf}}}{2}\left[L_2 \left|\lambda_2\right\rangle \left\langle T_+\right| + \mathbb{L}_2 \left|\lambda_2\right\rangle \left\langle T_-\right| + \mathrm{h.c.}\right]$$

$$L_2 = \nu_2 A_1^+ + \mu_2 A_2^+ \qquad \mathbb{L}_2 = \mu_2 A_1^- + \nu_2 A_2^-$$



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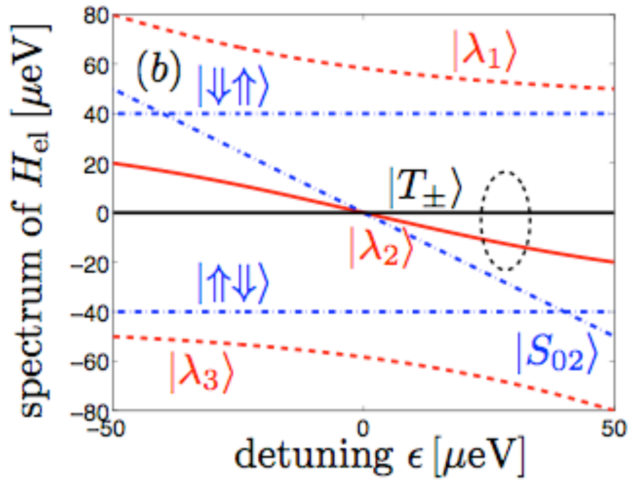
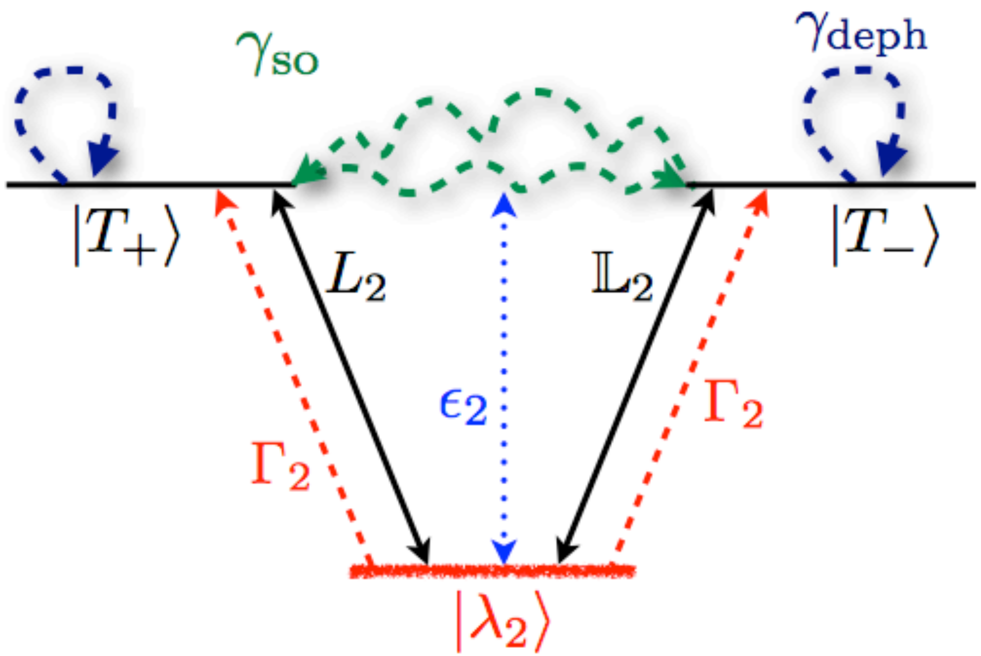



- Effective three-level dynamics



# Nuclear State Dynamics (2)

- Lindblad Master Equation (supplemental)

$$
\begin{aligned}
\dot{\rho} &= \mathcal{L}_0[\rho] + \mathcal{V}[\rho] \\
\mathcal{L}_0[\rho] &= -i[H_{\mathrm{el}}, \rho] + \Gamma_2 \sum_{\nu=\pm} \mathcal{D}[|T_\nu\rangle\langle\lambda_2|]\rho \\
&\quad + \gamma_{\mathrm{so}} \sum_{\nu=\pm} \mathcal{D}[|T_{\bar{\nu}}\rangle\langle T_\nu|]\rho + \mathcal{L}_{\mathrm{deph}}[\rho]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}[\rho] &= -i[H_{\mathrm{ff}} + H_{\mathrm{zz}}, \rho] \\
H_{\mathrm{zz}} &= a_{\mathrm{hf}} \textstyle\sum_{i=1,2} S_i^z \delta A_i^z
\end{aligned}
$$

# Nuclear State Dynamics (2)

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$$H_{\mathrm{zz}} = a_{\mathrm{hf}} \sum_{i=1,2} S_i^z \delta A_i^z$$

- If $\Delta \gtrsim 3\,\mu\mathrm{eV}$ and $\gamma_{\mathrm{so}}, \gamma_{\mathrm{deph}} \gg \sqrt{N} a_{\mathrm{hf}}$

$$\rho_{\mathrm{ss}}^{\mathrm{el}} = (|T_+\rangle\langle T_+| + |T_-\rangle\langle T_-|)/2$$

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- Lindblad Master Equation for Nuclear State

$$\dot{\sigma} = \mathcal{L}_{\mathrm{eff}}[\sigma] = \mathcal{L}_{\mathrm{id}}[\sigma] + \mathcal{L}_{\mathrm{nid}}[\sigma]$$

$$\sigma = \mathrm{Tr}_{\mathrm{el}}[\rho]$$

$$\begin{aligned}\mathcal{L}_{\mathrm{id}}[\sigma] &= \frac{\gamma}{2}[\mathcal{D}[L_2]\sigma + \mathcal{D}[\mathbb{L}_2]\sigma] \\ &\quad + i\frac{\delta}{2}\left(\left[L_2^\dagger L_2, \sigma\right] + \left[\mathbb{L}_2^\dagger \mathbb{L}_2, \sigma\right]\right)\end{aligned}$$

$$\begin{aligned}\gamma &= a_{\mathrm{hf}}^2 \tilde{\Gamma}/[2(\tilde{\Gamma}^2 + \epsilon_2^2)] \\ \delta &= (\epsilon_2/2\tilde{\Gamma})\gamma \\ \tilde{\Gamma} &= \Gamma_2 + \gamma_{\mathrm{so}}/2 + \gamma_{\mathrm{deph}}/4\end{aligned}$$

# Nuclear State Dynamics (2)

- Lindblad Master Equation (supplemental)

$$\dot{\rho} = \mathcal{L}_0[\rho] + \mathcal{V}[\rho]$$

$$\mathcal{L}_0[\rho] = -i[H_{\rm el},\rho] + \Gamma_2 \sum_{\nu=\pm} \mathcal{D}[|T_\nu\rangle\langle\lambda_2|]\rho + \gamma_{\rm so}\sum_{\nu=\pm}\mathcal{D}[|T_{\bar{\nu}}\rangle\langle T_\nu|]\rho + \mathcal{L}_{\rm deph}[\rho]$$

$$\mathcal{V}[\rho] = -i[H_{\rm ff} + H_{\rm zz}, \rho]$$

$$H_{\rm zz} = a_{\rm hf}\sum_{i=1,2} S_i^z \delta A_i^z$$

- If $\Delta \gtrsim 3\,\mu\mathrm{eV}$ and $\gamma_{\rm so}, \gamma_{\rm deph} \gg \sqrt{N} a_{\rm hf}$

$$\rho_{\rm ss}^{\rm el} = \left(|T_+\rangle\langle T_+| + |T_-\rangle\langle T_-|\right)/2$$

- Lindblad Master Equation for Nuclear State

$$\dot{\sigma} = \mathcal{L}_{\rm eff}[\sigma] = \mathcal{L}_{\rm id}[\sigma] + \mathcal{L}_{\rm nid}[\sigma]$$

$$|\xi_{\rm ss}\rangle = \sum_{k=0}^{2J}\xi^k|k, 2J-k\rangle$$

squeezing dynamics

$$\sigma = \mathrm{Tr}_{\rm el}[\rho]$$

$$\mathcal{L}_{\rm id}[\sigma] = \frac{\gamma}{2}\left[\mathcal{D}[L_2]\sigma + \mathcal{D}[\mathbb{L}_2]\sigma\right] + i\frac{\delta}{2}\left(\left[L_2^\dagger L_2, \sigma\right] + \left[\mathbb{L}_2^\dagger \mathbb{L}_2, \sigma\right]\right)$$

$$\gamma = a_{\rm hf}^2\tilde{\Gamma}/[2(\tilde{\Gamma}^2 + \epsilon_2^2)]$$

$$\delta = (\epsilon_2/2\tilde{\Gamma})\gamma$$

$$\tilde{\Gamma} = \Gamma_2 + \gamma_{\rm so}/2 + \gamma_{\rm deph}/4$$

# EPR Uncertainty for Testing Entanglement

*Theorem 1.*–Sufficient criterion for inseparability: For any separable quantum state $\rho$, the total variance of a pair of EPR-like operators defined by Eqs. (2a) and (2b) with the commutators $[\hat{x}_j, \hat{p}_{j'}] = i\delta_{jj'}$ $(j, j' = 1, 2)$ satisfies the inequality

$$\langle(\Delta\hat{u})^2\rangle_\rho + \langle(\Delta\hat{v})^2\rangle_\rho \geq a^2 + \frac{1}{a^2}$$

$$\hat{u} = |a|\hat{x}_1 + \frac{1}{a}\hat{x}_2, \qquad (2a)$$

$$\hat{v} = |a|\hat{p}_1 - \frac{1}{a}\hat{p}_2, \qquad (2b)$$

- Separability : $\rho = \sum_i p_i \rho_{i1} \otimes \rho_{i2}$

# Steady State Entanglement

- EPR Criteria : $\Delta_{\mathrm{EPR}} = \dfrac{\mathrm{var}\,(I_1^x + I_2^x) + \mathrm{var}\,(I_1^y + I_2^y)}{2\,(|\langle I_1^z \rangle| + |\langle I_2^z \rangle|)} < 1$

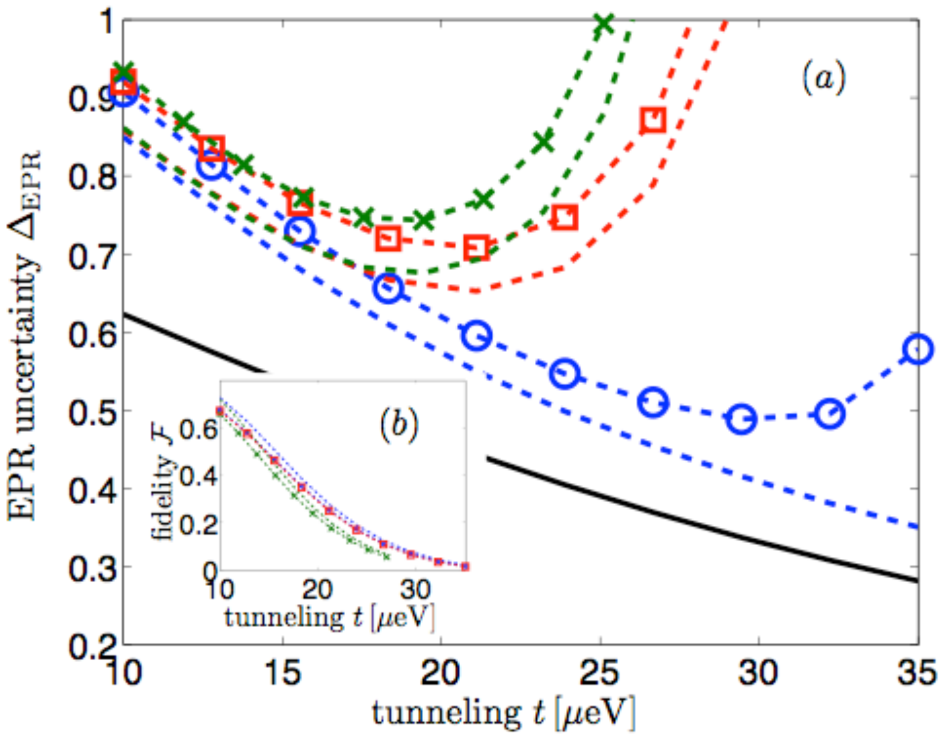


Figure 2: (color online). Steady-state entanglement between the two nuclear spin ensembles quantified via the EPR uncertainty $\Delta_{\mathrm{EPR}}$ (a) and fidelity $\mathcal{F}$ of the nuclear steady state with the two-mode squeezed target state (b); both shown here as a function of the interdot tunneling parameter $t$. As a benchmark, the black solid curve refers to the idealized setting where the undesired HF-coupling to $|\lambda_{1,3}\rangle$ has been ignored and where $J_1 = J_2 = pJ_{\max}$, $p = 0.8$ and $N_1 = N_2 = 2J_{\max} = 10^6$, corresponding to $\Delta_{\mathrm{OH}} = 40\mu\mathrm{eV}$. The blue-dashed line then also takes into account coupling to $|\lambda_{1,3}\rangle$ while the red-dashed curve in addition accounts for an asymmetric dot size: $N_2 = 0.8N_1 = 8 \times 10^5$. The amount of entanglement decreases for a smaller nuclear polarization: $p = 0.7$ (green dashed curve). Classical uncertainty (symbols) in the total spin $J_i$ quantum numbers leads to a reduced amount of entanglement, but does not disrupt it completely; here, we have set the range of the distribution to $\Delta_{J_i} = 50\sqrt{N_i}$. Other numerical parameters: $\omega_0 = 0$, $\Gamma = 25\mu\mathrm{eV}$, $\epsilon = 30\mu\mathrm{eV}$ and $\gamma_{\mathrm{so}} + \gamma_{\mathrm{deph}}/2 = 1\mu\mathrm{eV}$.

# Experimental Realization

- Entanglement is accompanied by generation of high gradient fields

$$\frac{d}{dt}\Delta_{I^z} = -\gamma_{\text{eff}}\left[\Delta_{I^z} - N\frac{\chi}{\gamma_{\text{eff}}}\right] \qquad \begin{aligned} \chi &= \gamma\left(\mu_2^2 - \nu_2^2\right)(3p_+ - 1) \\ \gamma_{\text{eff}} &= \gamma\left(\mu_2^2 + \nu_2^2\right)(1 - p_+)\,. \end{aligned}$$

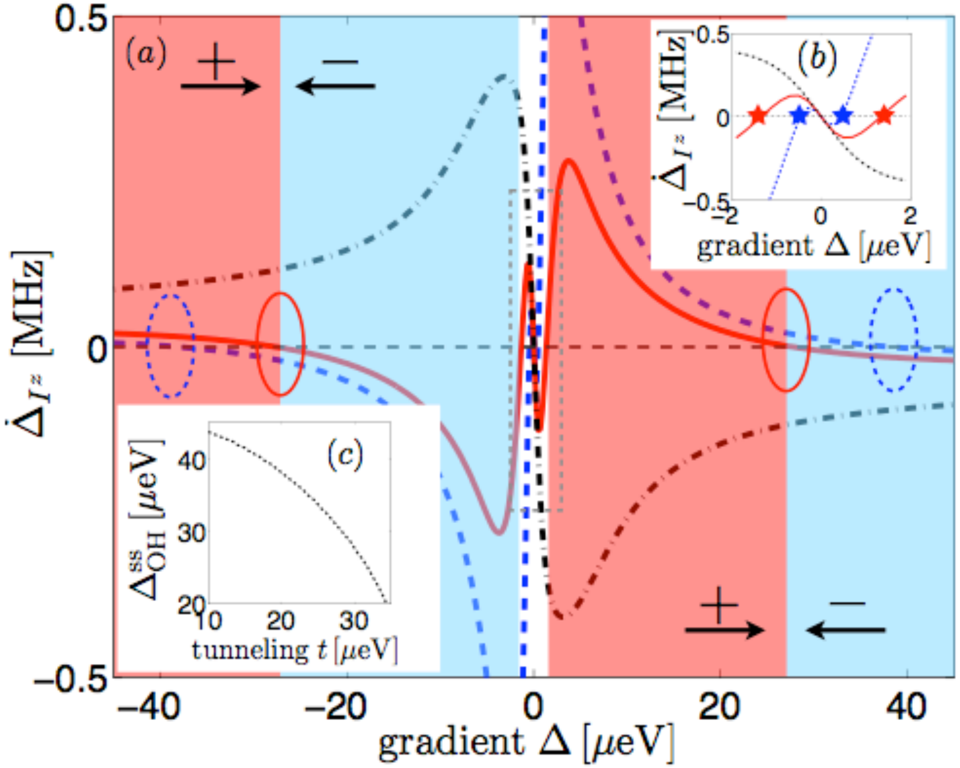


Figure 3: (color online). Semiclassical solution to the nuclear polarization dynamics. (a) Instantaneous nuclear polarization rate $\dot{\Delta}_{I^z}$ as a function of the gradient $\Delta$ for $t = 20\mu$eV (blue dashed), $t = 30\mu$eV (red solid) and $t = 50\mu$eV (black dash-dotted). Fixed points are found at $\dot{\Delta}_{I^z} = 0$. The ovals mark stable high-gradient steady state solutions. The background coloring refers to the sign of $\dot{\Delta}_{I^z}$ (shown here for $t = 30\mu$eV) which determines the stable fixed point the nuclear system is attracted to (see arrows). (b) Zoom-in of (a) into the low-gradient regime: The trivial, unpolarized fixed point lies at $\Delta = 0$, whereas critical, instable points $\Delta_{\text{OH}}^{\text{crt}}$ (marked by stars) can be identified with $\dot{\Delta}_{I^z} = 0$ and $d\dot{\Delta}_{I^z}/d\Delta > 0$. (c) Stable high-polarization fixed points $\Delta_{\text{OH}}^{\text{ss}}$ (see ovals) as a function of $t$; for $t \approx 10\mu$eV we obtain a nuclear polarization of approximately $\sim 90\%$. Other numerical parameters: $\Gamma = 25\mu$eV, $\epsilon = 30\mu$eV, $\gamma_{\text{so}} = 0.3\mu$eV and $\gamma_{\text{deph}} = 0.5\mu$eV.

# Does All of This Make Any Sense?

- Nuclear spins treated as stochastic random variable

- Initial nuclear state (already entangled?)

- Polarization dynamics treated within bosonic model (no correlation among nuclear spins)

- Reality: hyperfine coupling not homogeneous (generalization to inhomogeneous coupling requires identical dots)