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An area law for entanglement from exponential decay of correlations

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Area laws for entanglement in quantum many-body systems give useful information about their low-temperature behaviour and are tightly connected to the possibility of good numerical simulations. An intuition from quantum many-body physics suggests that an area law should hold whenever there is exponential decay of correlations in the system, a property found, for instance, in non-critical phases of matter. However, the existence of quantum data-hiding states—that is, states having very small correlations, yet a volume scaling of entanglement—was believed to be a serious obstruction to such an implication. Here we prove that notwithstanding the phenomenon of data hiding, one-dimensional quantum many-body states satisfying exponential decay of correlations always fulfil an area law. To obtain this result we combine several recent advances in quantum information theory, thus showing the usefulness of the field for addressing problems in other areas of physics.

Journal Club

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Relevance of groundstate properties

Properties of the ground state (gs) give a lot of information about the physics of a quantum many-body Hamiltonian at zero/low temperature:

- Decay of correlations allows to identify the phase of the model
- Entanglement in the gs: area laws

Relevance of gs entanglement

- Resource character of entanglement in QIP
- Signatures of criticality can be detected on the level of entanglement and quantum correlations
- Large amounts of entanglement in the gs render its classical simulation infeasible
- Conversely: limited entanglement allows for good methods for numerical simulation (e.g. Matrix Product State representation)

11. Horodecki, R., Horodecki, P., Horodecki, M. & Horodecki, K. Quantum entanglement. *Rev. Mod. Phys.* **81,** 865–942 (2009).

Measure of correlations

- For a bipartite (mixed) state ρ_{XY}

$$\mathrm{Cor}(X:Y) := \max_{\|M\|\leq 1, \|N\|\leq 1} |\mathrm{tr}((M \otimes N)(\rho_{XY} - \rho_X \otimes \rho_Y))|$$

- A pure state $|\psi\rangle_{1,\ldots,n}$ has ξ-exponential decay of correlations (EDC) if for any two regions X and Y we have

$$\mathrm{Cor}(X:Y) \leq 2^{-l/\xi}$$

where l=dist(X,Y).

Exponential decay of correlations

- … is expected to appear in non-critical phases of matter, where there is a notion of correlation length.
- It has been proved that ground states of gapped Hamiltonians have exponentially decaying correlations, where the correlation length is given by the inverse spectral gap.

Measure of entanglement

- For a bipartite pure state $|\psi\rangle_{XY}$

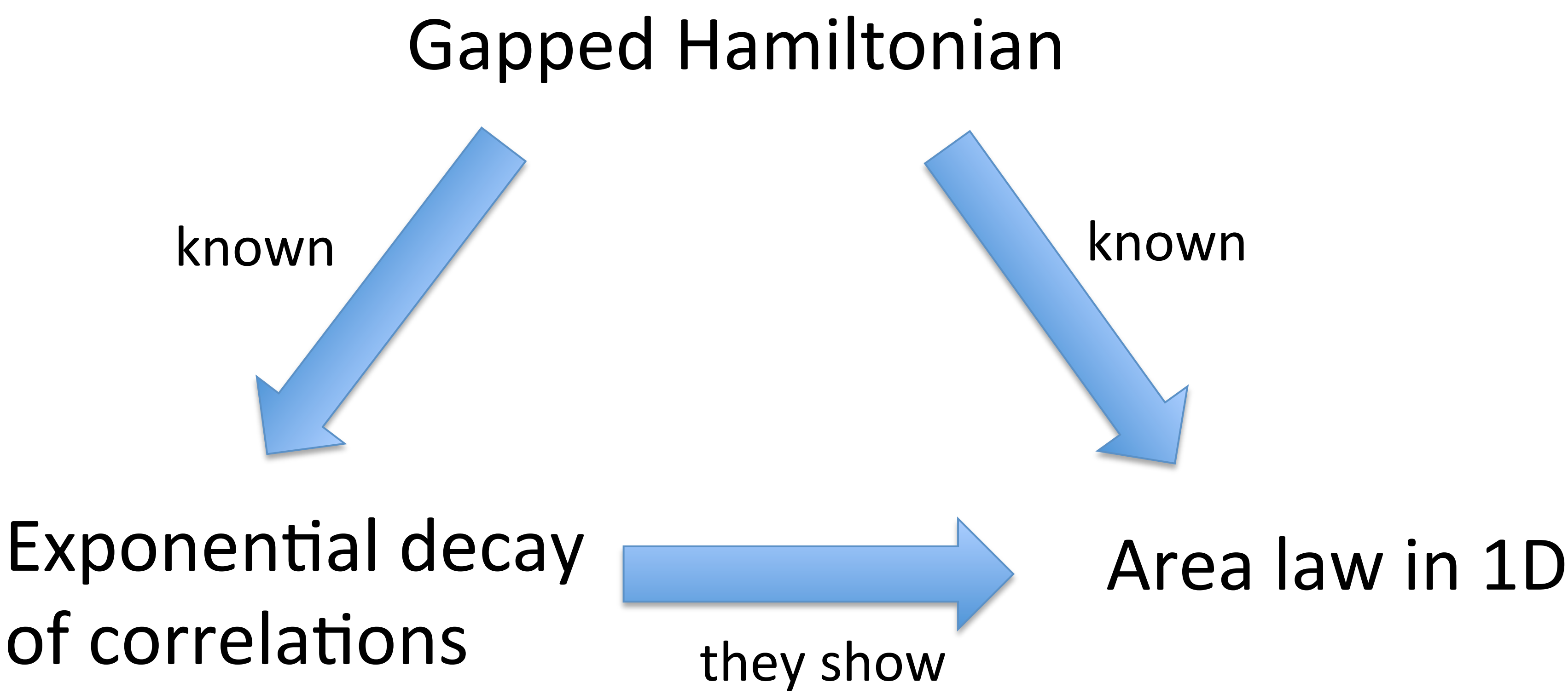
$$E(|\psi\rangle_{XY}) = H(\rho_X) = -\mathrm{tr}(\rho_X \log \rho_X)$$

- States corresponding to the ground or low-lying eigenstates of local models often satisfy area laws
  - Black holes
  - Quantum spin systems
  - Quantum harmonic systems

Area laws

- The entanglement of a contiguous region with its complement is proportional to the boundary, not its volume (as one would expect for a generic quantum state)
- M. Hastings (2007): 1D gapped Hamiltonians with a unique ground state always obey an area law (i.e., the entropy of a contiguous region is upper-bounded by a constant).

What is known and what they show



Intuition from many-body physics

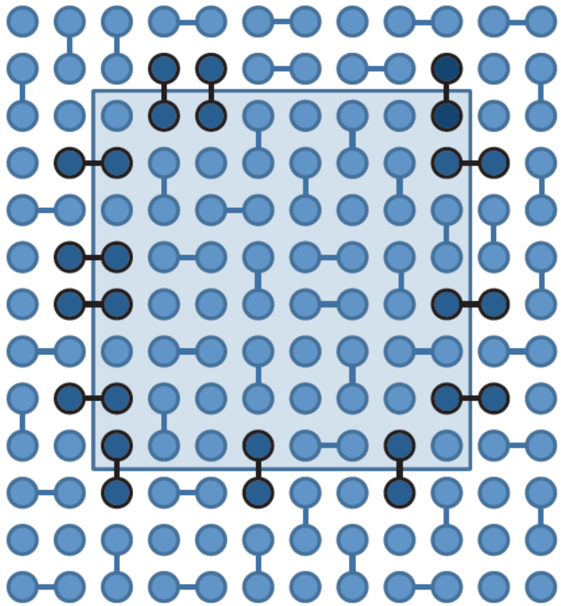


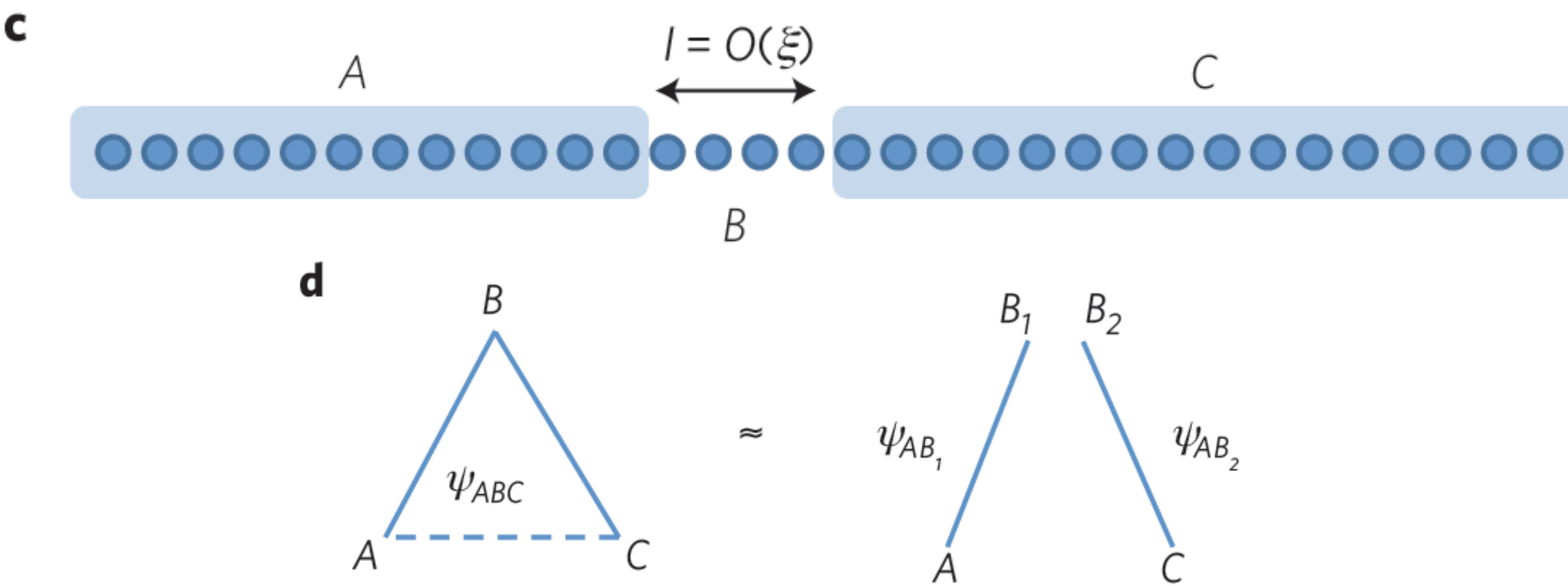
Figure 1 | EDC intuitively suggests an area law. a, The intuition is exemplified in a simple manner by a state consisting of entangled pairs of neighbouring particles. There the correlations are of fixed length 2, as only neighbours are correlated. The particles connected by an edge are in the pure state $\psi = (1/2)(|00 + |11)$, and so only the pairs crossing the boundary (dark blue) contribute to the entropy of the region inside the boundary (shaded square). **b**, For 1D states an area law implies that the

Intuition from many-body physics

b

boundary (shaded square). **b**, For 1D states an area law implies that the entropy of an interval is constant. Again for a system of entangled pairs, only one pair cut the boundary. **c**,**d**, A general intuitive argument is the

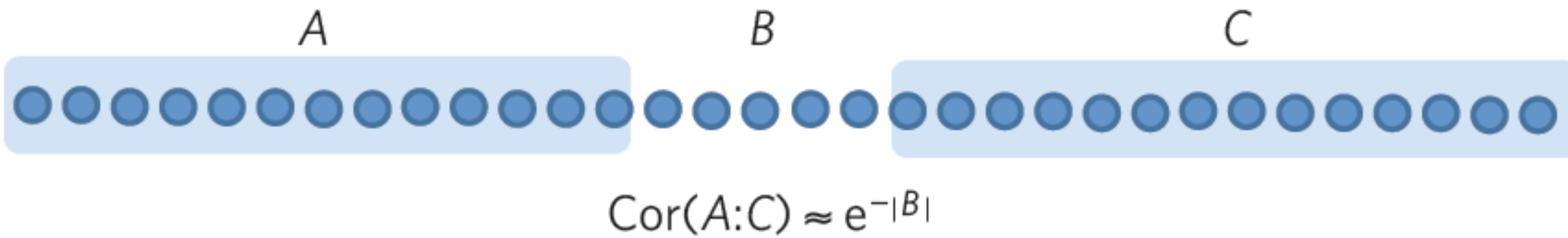
Intuition from many-body physics



only one pair cut the boundary. **c**,**d**, A general intuitive argument is the following: if the distance of two parts A and C is larger than the correlation length, the reduced state ρ_{AC} should be close to a product state: $\rho_{AC} \approx \rho_A \otimes \rho_C$, then suggesting that the system B can be divided into subsytems B_1 and B_2 such that the total pure state ψ_{ABC} is close to the product state $\psi_{AB_1} \otimes \psi_{AB_2}$. However, for pure bipartite states, the entropy cannot exceed the size of any of subsystems. Therefore, $S(A) \leq S(B_1) \approx O(\xi)$, and we would obtain that entropy of any interval is constant and proportional to the correlation length ξ.

The flaw in the argument

Data hiding: For almost all states $|\psi\rangle_{ABC}$ with $|A| \approx |C|$, the correlations between A and C are vanishing but the entropies of A and C are close to maximal.



Quantum data-hiding states have been thought of being an obstruction for obtaining an area law for entanglement entropy from EDC.

Main result

***Theorem* 1.** Let $|\psi\rangle_{1,\ldots,n}$ be a state defined on a line with ξ-EDC. Then for any connected region $X \subset [n]$,

$$H(X) \leq c \exp\left(c' \log(\xi)\xi\right)$$

with $c, c' > 0$ being universal constants.

Application to many-body physics

- Ground states of (disordered) Hamiltonians exhibiting many-body localization (or a mobility gap) satisfy EDC.
- These models are in general not gapped, so no area law was known before.
- E.g., the XY model with random coefficients exhibits many-body localization ➔ its ground state satisfies an area law.

Application to MPS

- Matrix product state: description of quantum states, which is efficient in many physical contexts
- $$|\psi\rangle_{1,\dots,n} = \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} \mathrm{tr}(A_{i_1}^{[1]} \cdots A_{i_n}^{[n]}) |i_1, \dots, i_n\rangle$$

  where $A^{[i]}$ are DxD matrices
- D measures the complexity of the matrix product representation; when D=poly(n) the quantum state psi admits an efficient classical description ➔ EV's of local observables can be calculated efficiently.
- This is the case if psi satisfies EDC.

Application to MPS

Thus, 1D pure quantum states with EDC have a very simple structure, admitting a classical efficient parametrization. In fact, the most successful numerical method known for computing low-energy properties of 1D models, the density matrix renormalization group[15], is a variational method over the class of MPS. Corollary 2 shows that one should expect the density matrix renormalization group to work well whenever the model is such that its ground state has rapidly decaying correlations.

Application to QIP

- Which properties are behind the (apparent) superiority of quantum computation over classical compuation?
- ➔ Find conditions under which quantum circuits have an efficient classical simulation.
- Most famous result: Gottesman-Knill Theorem.

Application to QIP

***Corollary* 3.** Consider a family of quantum circuits $V = V_k \cdots V_2 V_1$ acting on n qubits arranged in a ring and composed of two qubit gates V_k. Let $|\psi_t\rangle := V_t \cdots V_2 V_1 |0^n\rangle$ be the state after the t th gate has been applied. Then if there is a constant ξ independent of n such that, for all n and $t \in [n]$, $|\psi_t\rangle$ has ξ-EDC, one can classically simulate the evolution of the quantum circuit in poly(n, k) time.

➔ If a quantum circuit is supposed to solve a classically hard problem more efficiently, one must have at least algebraically decaying correlations (cf. critical phases of matter).

Proof of main theorem

- Tools from Quantum Information Theory:
  - Quantum state merging
  - Single-shot quantum information theory

Open questions

- Linear rather than exponential bound (in ξ) in main Theorem.
- Generalization to higher dimensions. In 2D, it is not even known whether a gapped local Hamiltonian implies an area law.

Conclusions

- EDC implies an area law in 1D; these are properties of the state alone, no Hamiltonian enters the theorem.
- Applications to
  - Many body physics (disordered 1D systems)
  - Efficient description of quantum states (MPS)
  - Quantum Information Processing
- Usefulness of quantum information theory for addressing problems in other areas of physics.