Time-Reversal-Invariant Z_4 Fractional Josephson Effect

Fan Zhang and C. L. Kane - arXiv:1404.1072

We study the Josephson junction mediated by the quantum spin Hall edge states and show that electron-electron interactions lead to a dissipationless fractional Josephson effect in the presence of time-reversal symmetry. Surprisingly, the periodicity is 8π, corresponding to a Josephson frequency $eV/2\hbar$. We estimate the magnitude of interaction induced many-body level splitting responsible for this effect and argue that it can be measured using tunneling spectroscopy. For strong interactions we show that the Josephson effect is associated with the weak tunneling of charge $e/2$ quasiparticles between the superconductors. Our theory describes a fourfold ground state degeneracy that is similar to that of coupled "fractional" Majorana modes, but is protected by time reversal symmetry.

April 8, 2014

Layout

(a)

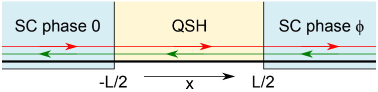
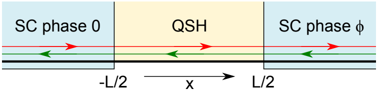


Figure: (a) Schematic of a Josephson junction mediated by the QSH edge states.

(a) 

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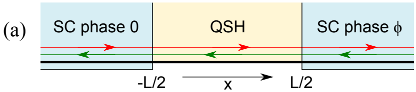
- $\mathcal{H}_{BdG} = \tau^z(-i\hbar v_F \sigma^z \partial_x - \mu) + \Delta_1(x)\tau^x + \Delta_2(x)\tau^y$,



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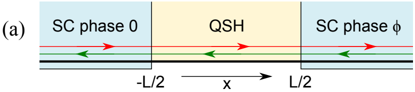
- $\vec{\sigma}$ ($\vec{\tau}$) are Pauli matrices in spin (particle-hole) space and $\Delta = \Delta_1 + i\Delta_2$ is the proximity induced pair potential



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- $\Delta(x < -L/2) = \Delta_0$, $\Delta(x > L/2) = \Delta_0 e^{i\phi}$, and $\Delta(|x| < L/2) = 0$

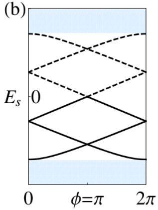
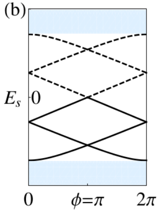


Figure: (b) The single-particle BdG spectrum of the junction as a function of ϕ, with Kramers degeneracies at $\phi = 0$ and π.



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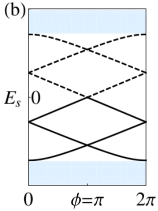

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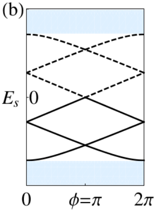
- $L > hv_F/(4\Delta_0) \Rightarrow$ at least one pair of excited bound states

Single particle spectrum



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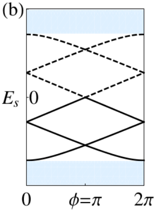
- $$\psi_{n,\sigma} = \begin{pmatrix} u_{n,\sigma} \\ v_{n,\sigma} \end{pmatrix} = \mathcal{A}_n(x) \begin{pmatrix} (-1)^n e^{i\sigma \bar{E}_n \bar{\ell}(x)} \\ -i\sigma e^{-i\sigma \bar{E}_n \bar{\ell}(x)} \end{pmatrix},$$
  $$\mathcal{A}_n(x) = 1\sqrt{2L + 2\xi(1-\bar{E}_n^2)^{-1/2}} e^{i\sigma\bar{\mu}\bar{x} - \sqrt{1-\bar{E}_n^2}|\bar{x}-\bar{\ell}(x)|}$$



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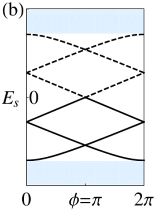
- $\bar{E}_n \equiv E_n/\Delta_0$, $\bar{L} \equiv L/\xi = L\Delta_0/\hbar v_F$, $\bar{\ell}(x) = sgn(x)\bar{L}/2$ in central region, and x/ξ elsewhere, $\bar{\mu} = \mu/\Delta$, $\bar{x} = x/\xi$



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- $E_{-n} = -E_n$ and $\psi_{-n,\sigma} = -i\tau^y \psi_{n,\sigma}$ $(n > 0)$

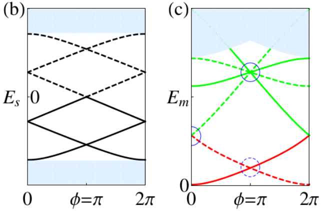
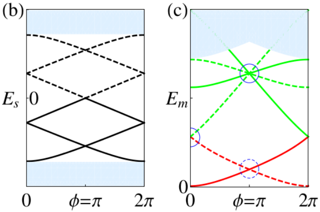


Figure: (c) The spectrum of many-body states corresponding to (b), including a fourfold degeneracy at $\phi = \pi$.

- The lowest state corresponds to the many-body ground state with all positive (negative) energy single-particle states in (b) empty (occupied), whereas higher states are excitations with one or more quasiparticles excited.



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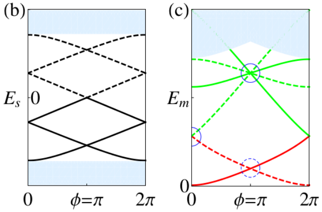
- ABS: $b_{-n,\sigma} = \sigma b^{\dagger}_{n,-\sigma}$ $(n > 0)$ and $b_{0,+} = i b^{\dagger}_{0,-}$



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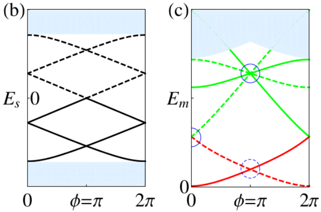

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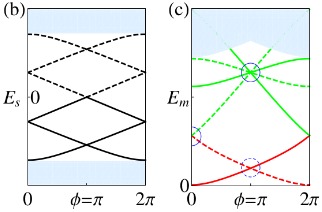
- $\mid \Phi_n(\phi)\rangle = b_N^{\dagger n_N} \dots b_1^{\dagger n_1} \mid 0\rangle$, $n_m = \{0, 1\} \equiv$ occupation probability of mth ABS



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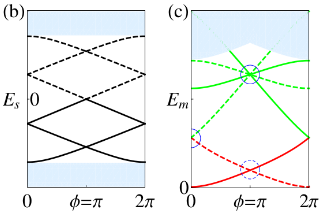

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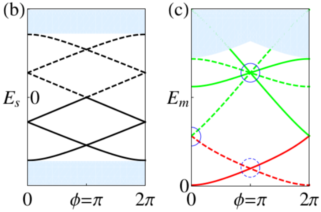

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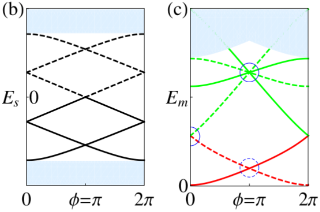

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- Solid and dashed $\equiv$ different local fermion parity

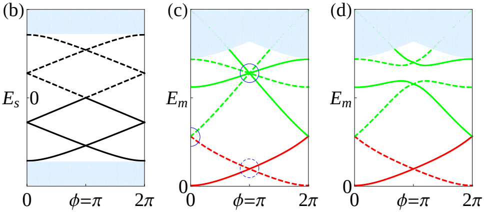
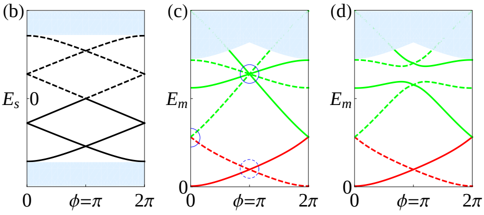


Figure: (d) The fourfold degeneracy is lifted by electron-electron interactions, leading to an 8π periodicity of the four lowest states.



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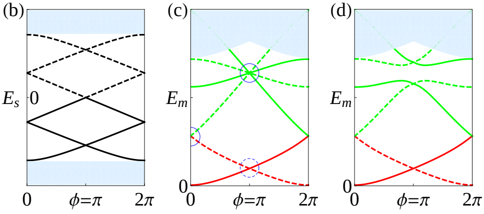
- In the presence of electron-electron interactions it splits into two Kramers doublets, each of which has two many-body states with opposite fermion parity.



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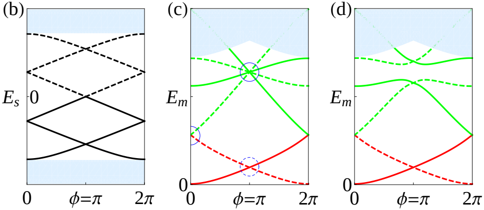
- In the presence of electron-electron interactions it splits into two Kramers doublets, each of which has two many-body states with opposite fermion parity.
- It takes four cycles to return to the original ground state, leading to an 8π periodicity in the current phase relation



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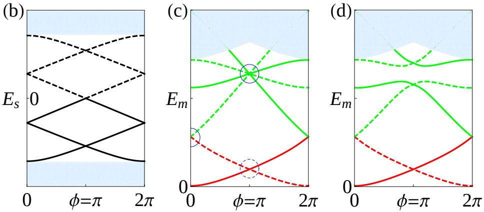
- $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$, $\mathcal{H}_I = \lambda \int_{-L/2}^{L/2} n(x)^2$



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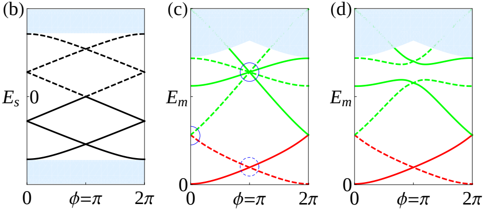

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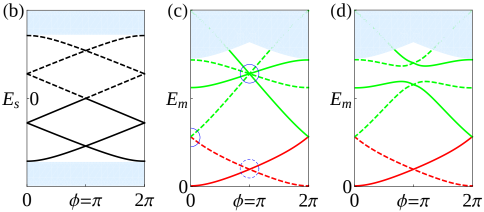

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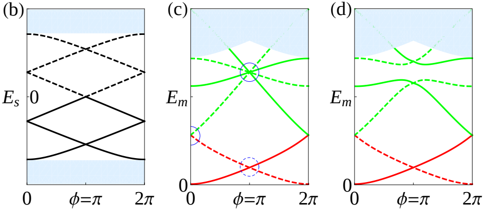

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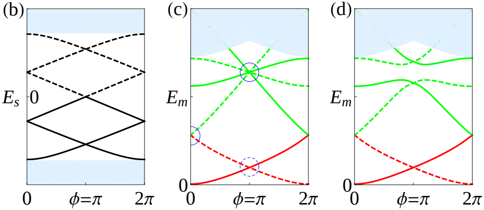

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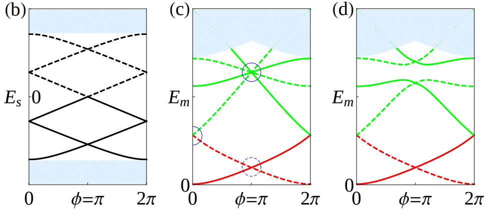

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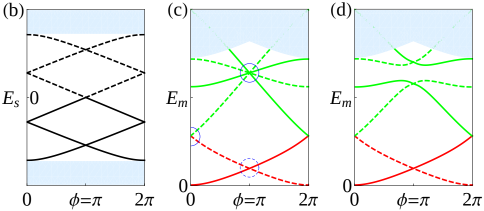
- $\mu = (-1)^{b^\dagger_{0,+} b_{0,+}}$ is the fermion parity



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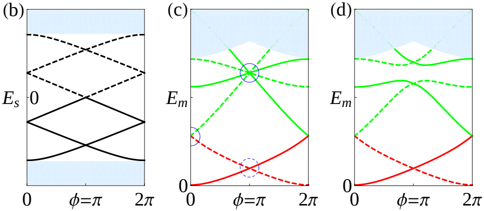
- At $\phi = \pi$, the splitting due to $\mathcal{H}_I$ becomes
$$\delta \sim 2\lambda \int_{-L/2}^{L/2} dx |u_{1,+}^* u_{0,-} - i u_{-1,-} u_{0,+}^*|^2 = \frac{\lambda}{\xi}\left(\frac{1}{\sqrt{1-\bar{E}_1^2}} + \bar{L}\right)^{-1}$$



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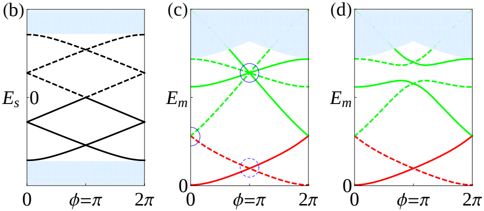
- $\bar{L} \sim 2.6$, $\delta \sim 0.23\lambda/\xi$, $\lambda = (e^2/\epsilon)\log(R_s/R)$



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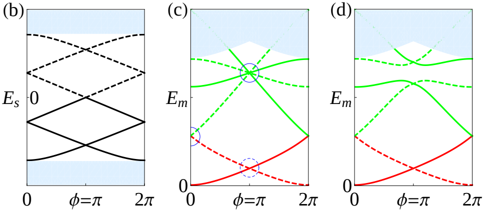
- $R, R_s \equiv$ penetration and screening radius of the edge states



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- $R, R_s \equiv$ penetration and screening radius of the edge states
- $\epsilon = 20, \xi = 100$ nm, and $\log(R_s/R) = 1 \Rightarrow \delta \sim 0.17$meV

Layout

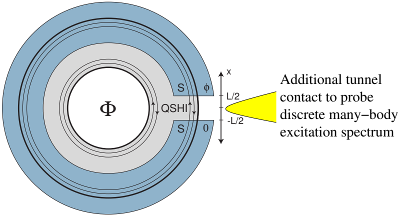
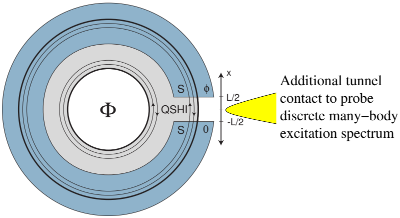


Figure: At low temperatures, weakly coupled tunnel junction probes the local tunneling density of states $dI/dV \propto \rho(E = eV)$



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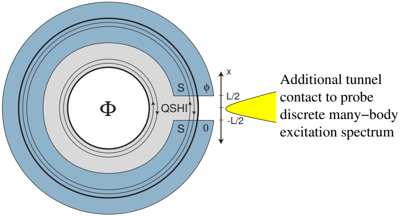
- $\rho(E) = \sum_{N,\sigma} |\langle N|c^\dagger_\sigma|0\rangle|^2 \delta(E - E_N + E_0)$



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- $\rho(E) = \sum_{N,\sigma} |\langle N|c_\sigma^\dagger|0\rangle|^2 \delta(E - E_N + E_0)$
- $c_\sigma^\dagger$ is the creation operator for an electron with spin σ in the junction and $|N\rangle$ are the many-body states

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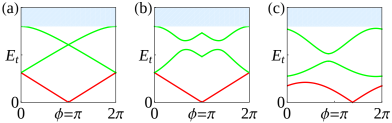

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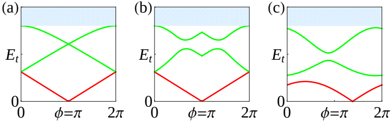

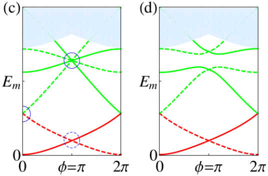

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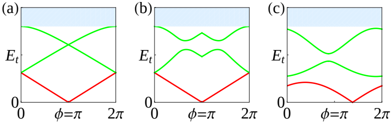

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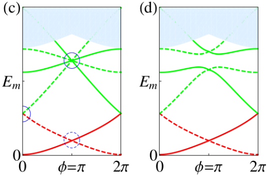

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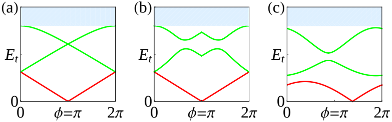

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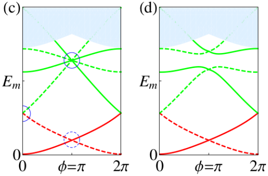

- Singularity lowest peak goes to zero
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Layout

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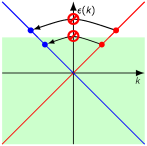

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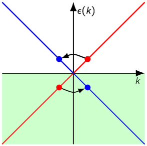

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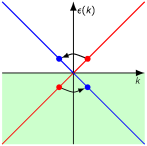

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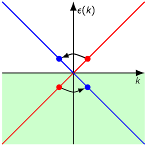

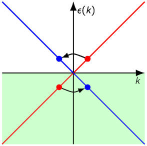

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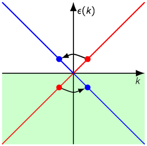

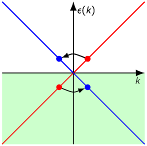

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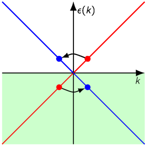

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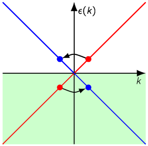

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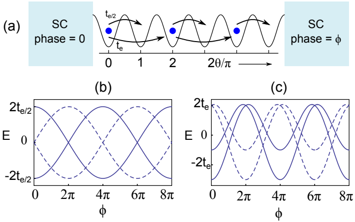
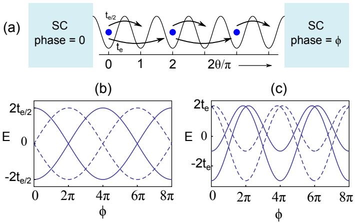
- For large v_0, θ is pinned in *four* distinct deep wells of the cosine potential
- Quantum tunneling between the 4 minimas (finite v_0, T) will couple the ground states, lifting their degeneracy with a characteristic pattern

Hamiltonian in the degenerate ground state



(a) Strong interactions pin the charge between the superconductors and lead to a fourfold ground state degeneracy. Charge $e/2$ or charge e tunneling processes lift the degeneracy, with an 8π periodicity in ϕ, as shown in (b) for $t_e = 0$ and (c) for $t_e = 2t_{e/2}$. Solid and dashed lines correspond to states with opposite fermion parity.



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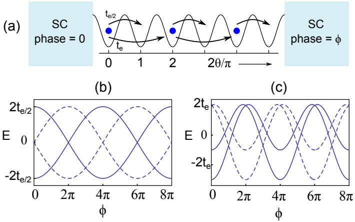

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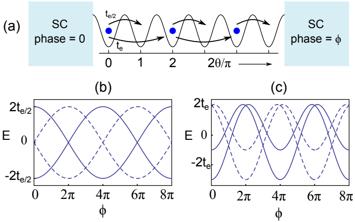

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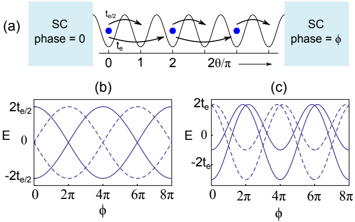

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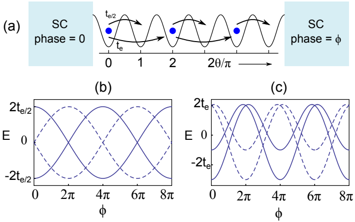

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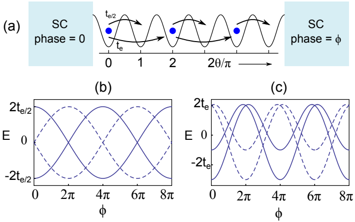

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- Thank you: Comments + Questions ???