Journal Club James Wootton 6^{th} May 2013

Violation of the Arrhenius law below the transition temperature

Beni Yoshida
Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, California 91125, USA
(Dated: April 3, 2014)

When interacting spin systems possess non-zero magnetization or topological entanglement entropy below the transition temperature, they often serve as classical or quantum self-correcting memory. In particular, their memory time grows exponentially in the system size due to polynomially growing energy barrier, as in the 2D Ising model and 4D Toric code. Here, we argue that this is not always the case. We demonstrate that memory time of classical clock model (a generalization of ferromagnet to q-state spins) may be polynomially long even when the system possesses finite magnetization. This weak violation of the Arrhenius law occurs above the percolation temperature, but below the transition temperature, a regime where excitation droplets percolate the entire lattice, yet the system retains a finite magnetization. We present numerical evidences for polynomial scaling as well as analytical argument showing that energy barrier is effectively suppressed and is only logarithmically divergent. We also suggest an intriguing possibility of experimentally observing the precession of magnetization vectors at experimentally relevant time scale.

Self-Correcting memories

- This paper considers self correcting memories
- Typically, these are interacting spin systems with a degenerate ground state
- Information (classical or quantum) is stored in the ground states
- System is then left for time t, subject to thermal errors
- System is then measured
- The final state is used to guess the initial state, and so retrieve stored info
- Lifetime of the memory is defined such that retrieval works with almost certainty when

$$t < \tau$$

- System is considered to be self-correcting when

$$\tau \geq O(L^c)$$

Energy Barrier

- This paper considers the relationship between the lifetime and the energy barrier

- This is defined using a set of single spin operators $\{U_t\}$ such that

$$\prod_t U_t |0\rangle = |1\rangle$$

- The energy after only the first *T* operations are applied is E_T

$$\prod_t^T U_t |0\rangle$$

- The energy barrier is the maximum energy during this process, minimized over all processes

$$E_B = min_{\{U_t\}} max_T E_T$$

Arrhenius Law

The memory time below T_c can be estimated roughly by

$$\tau \sim \exp\left(\frac{E_B}{K_B T}\right) \quad (1)$$

where E_B is an energy barrier between two degenerate ground states and T is the temperature. This predicts exponentially diverging memory time $\tau \sim \exp(L)$ at low temperatures, and thus the Ising model is a self-correcting memory. This empirical formula in Eq. (1) is often called the Arrhenius law and is widely used to predict time scales of stochastic processes such as chemical reaction time.

- Energy barrier can be used to get a rough idea of the lifetime
- Entropy will also play a role, but the 'law' restricts to ordered systems to avoid this
- A quick Google search suggests that it is only the author who uses this name regularly

Example: Ising model

- Ising model has two state classical spins
- Hamiltonian wants nearest neighbours to align
- Two ground states, all up and all down, so can store a bit
- 1D Ising model

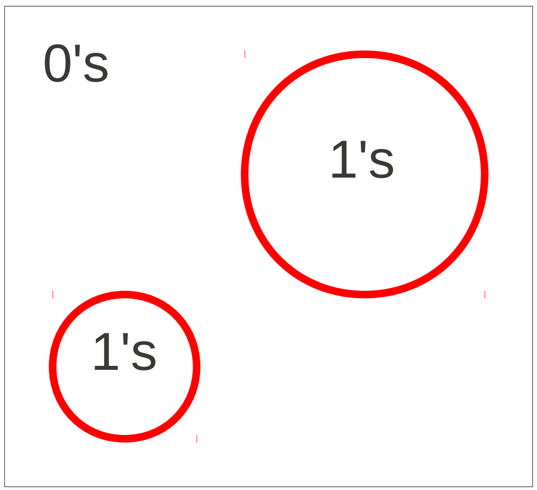
$\tau = O(1)$ $\qquad E_B = O(1)$ $\qquad$ 0000111000000011110000

- >1D Ising model: L^D lattice

$$E_B = O(L)$$

$$\begin{aligned} \text{magnetic order } (T < T_c) \quad &\Rightarrow \quad \tau \sim \exp(L^{D-1}) \\ \text{no order } (T > T_c) \quad &\Rightarrow \quad \tau \sim \log(L) \end{aligned}$$

where $D > 1$ is the spatial dimension. Indeed, the above relations have been proven mathematically for $D = 2$ [16] and confirmed for $D = \infty$ in the mean field model.



Clock models

- Generalization of Ising model to *q*-state classical spins
- Nearest neighbours want to align
- The more displaced they are, the more energy they pay

Clock model – The q-state clock model is a generalization of the Ising model to q-state spins. Spin values $s_j = 0, \cdots, q-1$ may be associated with angular variables $\theta_j = \frac{2\pi}{q} s_j$. The Hamiltonian of the clock model is

$$H = -\sum_{[i,j]\in n.n.} \cos(\theta_i - \theta_j) \qquad (2)$$

The system has q degenerate ground states which are separated by $O(L)$ energy barrier. The total magnetization m and the total angular value θ is given by $me^{i\theta} = \frac{1}{L^2}\sum_j m_j$ where $m_j \equiv e^{i2\pi s_j/q}$.

Phase Transitions

- For q=2,3,4 there is a finite temperature phase transition
- *m* is finite for $T < T_c$, but vanishes for $T > T_c$
- For higher q there are two transitions
- *m* is finite for $T < T_c$
- majority spin value percolates for

$$T < T_p < T_c$$

- System is magnetically ordered below T_c, so Arrhenius law would seem to predict

$$\tau \sim e^{\beta E_B}$$

for both the low and medium T regimes

- Is this true?

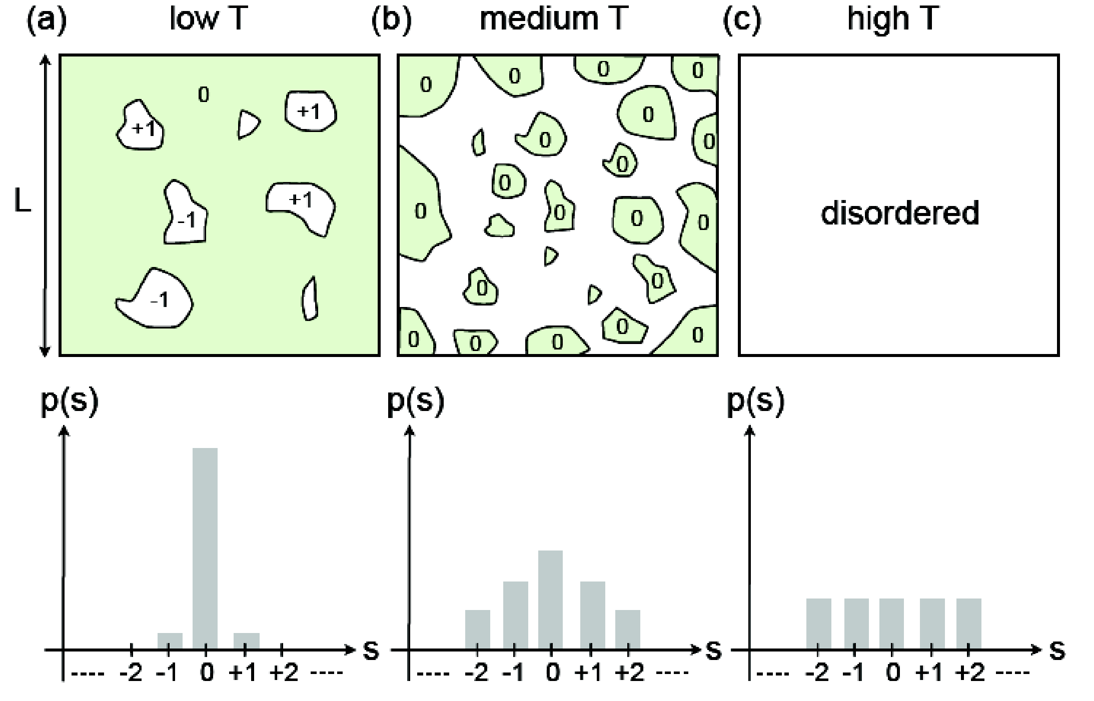


FIG. 1: Static properties of the clock model. (a) Low temperature phase. (b) Intermediate temperature phase. (c) High temperature phase.

Lifetime for medium T

The plots can be fitted by straight lines, implying that the memory time is indeed polynomially diverging, with $\tau \propto L^z$.

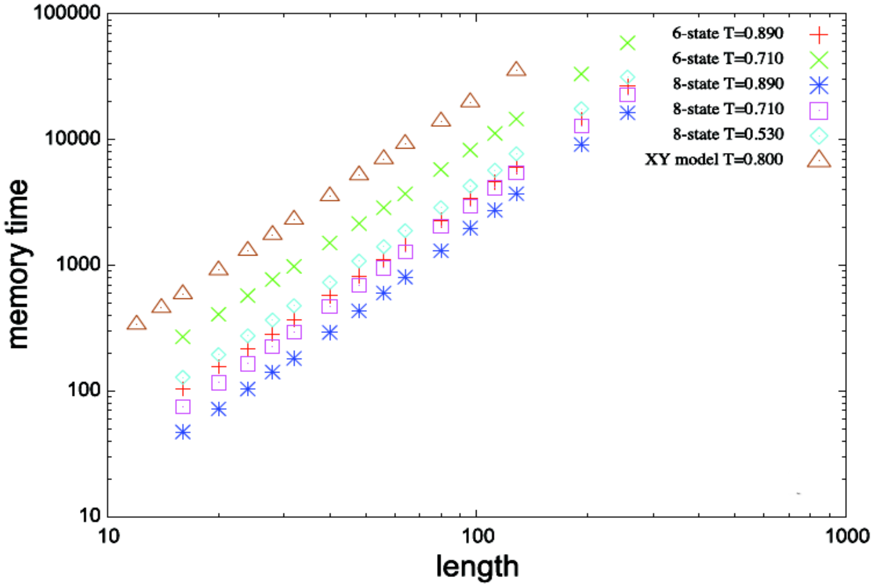


FIG. 2: Memory time of the clock model and the XY model versus length L in a log-log plot. All the temperatures are in the intermediate regime.

Estimates at temperatures near T_p all give $z \approx 2.0$ for $q \geq 6$.

state q	temperature	exponent z
2	T_c	2.12(1)
3	T_c	2.11(1)
5	0.950	1.92(1)
5	0.910	1.92(2)
6	0.890	2.11 (1)
6	0.710	2.00 (1)
8	0.890	2.13 (2)
8	0.710	2.07 (1)
8	0.530	2.04 (1)
8	0.430	1.98 (2)
12	0.890	2.61(5)
12	0.550	2.30(8)
12	0.200	1.99 (3)
∞	0.800	1.96 (1)

TABLE I: Exponents of memory time for the clock model and the XY model.

- Lifetime increases quadratically, not exponentially
- Does this violate the law?

Energy Barriers 2

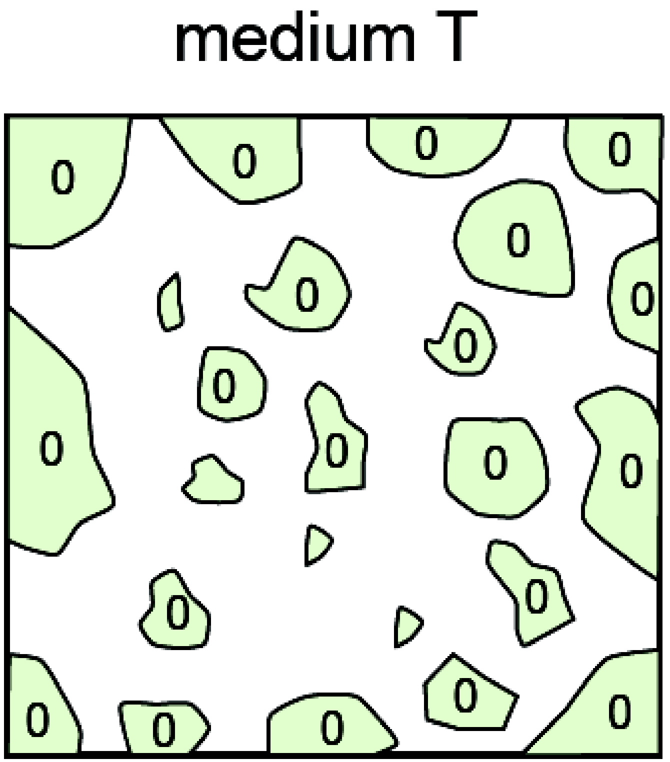
- To solve the apparent violation, energy barriers are redefined

- Rather than going ground state to ground state

$$|0\rangle \rightarrow |1\rangle \qquad \prod_t U_t |0\rangle = |1\rangle$$

we consider going from a typical thermal state with one spin value in the majority, to a state with another majority

$$\rho_0 \rightarrow \rho_1$$



- Lack of percolation means isolated clusters of size not greater than *~log(L)*

- Energy barrier dominated by flipping the largest cluster

$$E_B = O(\log L)$$

- This is consistent with the lifetime result

$$\tau = e^{O(\log L)} = \mathrm{poly}(L)$$

XY model

- Infinite q corresponds to the XY model

$$T_p = 0 \quad T_c = O(1)$$

- This also has poly(L) lifetime
- However, *m* is only finite when q is finite. It vanishes for finite T in the XY model
- So this is an example of a poly(L) lifetime in the absence of order

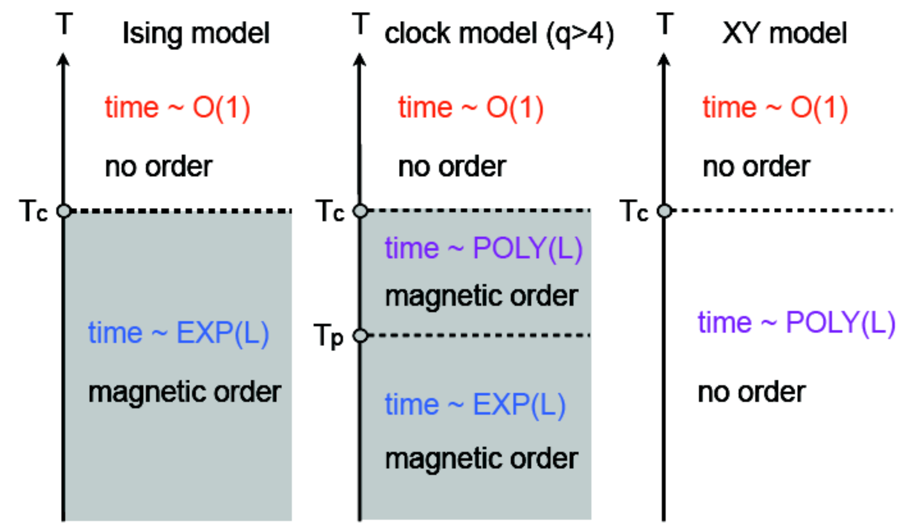


FIG. 3: Time scales of the clock model. Shaded regions represent phases with magnetic order. The system possesses magnetic order and polynomially diverging (experimentally relevant) time scale only for intermediate values of q.

Conclusion

In summary, the following relations seem to hold:

$$\begin{aligned} \text{magnetic order } (T < T_c) \quad &\Rightarrow \quad \tau \geq \text{poly}(L) \\ \text{no order } (T \geq T_c) \quad &\Rightarrow \quad \tau \leq \text{poly}(L) \end{aligned} \tag{3}$$

The q-state clock model ($q \geq 5$) and two-dimensional XY model ($q \to \infty$) correspond to the equalities in the above relations.

- Conclusion is that order seems to imply self-correction, though not always as good as we'd like

- Disorder doesn't always imply that self-correction is impossible

Outlook

- Time evolution of spin occurs on experimentally relevant timescale
- Might be interesting to observe
- multiferroic hexagonal manganites could provide q=6

The system has q degenerate ground states which are separated by $O(L)$ energy barrier. The total magnetization m and the total angular value θ is given by $me^{i\theta} = \frac{1}{L^2}\sum_j m_j$ where $m_j \equiv e^{i2\pi s_j/q}$.

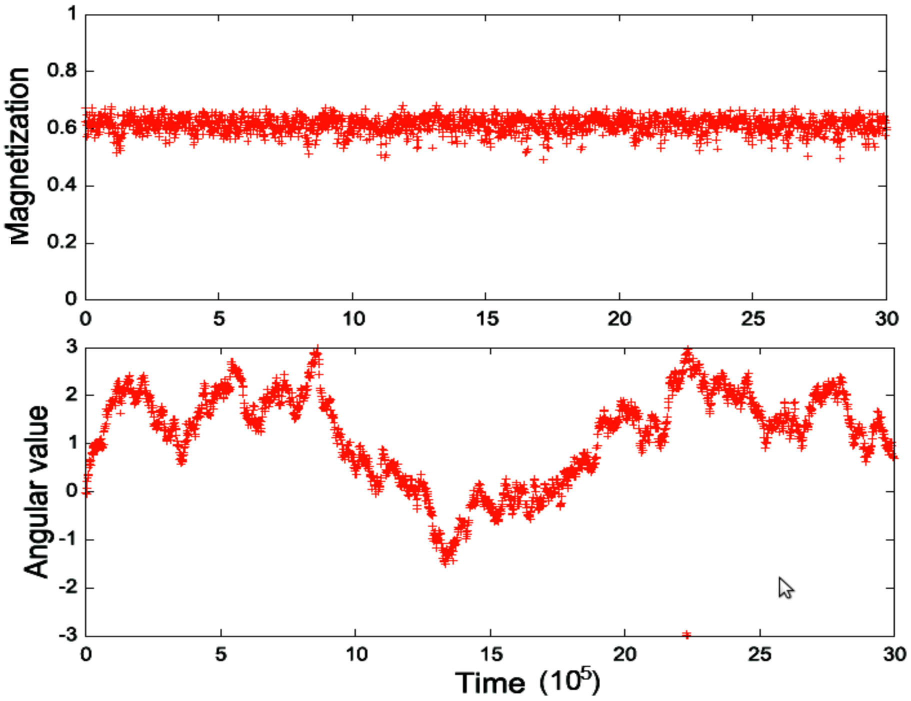


FIG. 4: Time evolutions of magnetization and polarization angle for $3 \cdot 10^6$ Monte-Carlo steps. Polarization angles are converted into effective spin values $\theta/2\pi$ modulo 6.

Thanks for your attention