Cellular-automaton decoders for topological quantum memories

M. Herold,[1] E. T. Campbell,[1,2] J. Eisert,[1] and M. J. Kastoryano[1]

[1] *Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany*

[2] *Department of Physics and Astronomy, University of Sheffield, Sheffield S3 7RH, UK*

We introduce a new framework for constructing topological quantum memories, by recasting error recovery as a dynamical process on a cellular automaton. We envisage quantum systems controlled by a classical hardware composed of small local memories, communicating with neighbors, and repeatedly performing identical simple update rules. This approach does not require any global operations or complex decoding algorithms. Our cellular automata draw inspiration from classical field theories, with a Coulomb-like potential naturally emerging from the local dynamics. For a 3D automaton coupled to a 2D toric code, we present evidence of an error correction threshold above 6.1% for uncorrelated noise. A 2D automaton equipped with a more complex update rule yields a threshold above 8.2%. Our framework provides decisive new tools in the quest for realizing a passive dissipative quantum memory.

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Motivation

- Prolonging the lifetime of quantum information is a necessary first step towards scalable quantum computing and quantum communication => need quantum error correction (QEC)
- Most promising: topological codes (paradigmatic example: toric code)
- QEC requires to continuously perform syndrome measurements and a classical decoding algorithm
- The decoding process needs to be much quicker than the decoherence time
- QEC typically requires rapid communication between many spatially separated (classical) cores

Their idea

- Perform QEC by means of a cellular automaton decoder
- Cellular automaton: network of cells with local update rules
- Rich dynamics, Turing complete (*e.g.* Game of Life)
- Mediate long-range attractions between excitations (“gravitational field”)

Active error correction:

- Continuously extract syndrome information from the code
- Use a classical algorithm to decode it (non-local information processing necessary) *
- Undo the errors

Passive error correction:

- Couple to topological code to an auxiliary (quantum) system
- This coupling "protects" the topological code
- No classical computation or active correction necessary

Cellular automaton decoder:

- Only simple, *local*, classical update rules necessary

*** disputed**

Toric code, error strings, anyons

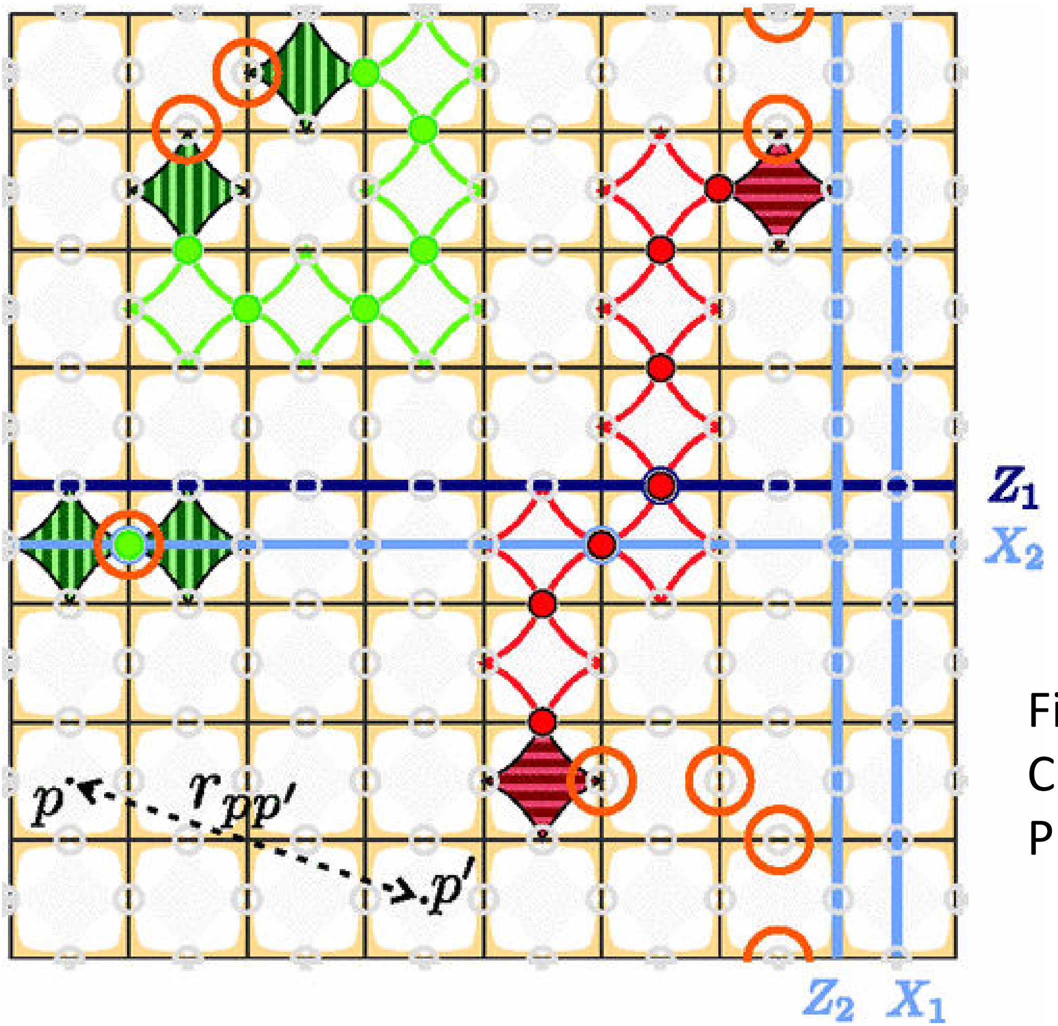
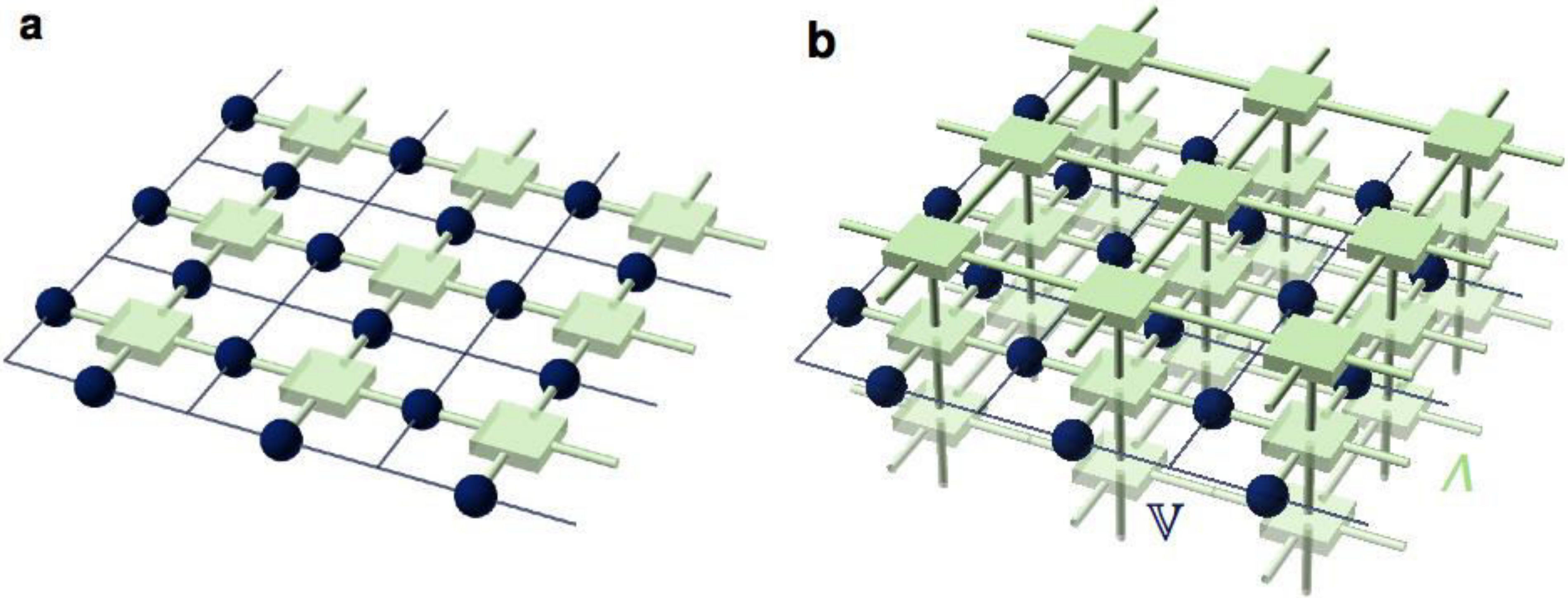


Figure borrowed from:
Chesi, Roethlisberger, and Loss,
PRA 2010

Active error correction

- Consider a code of size LxL and iid X-errors with probability p
- There is a threshold error rate p_c=10.3% such that
  - For $p<p_c$ an efficient classical decoder can decode the syndrome (= anyon locations) such that the failure probability is exponentially small in L
  - For $p>p_c$, the chances of successful decoding approach ½ for large L

Illustration of code and automata



2D automaton

3D automaton

The automaton

- Interplay between two fields $q_E(\mathbf{x})$ (anyonic charge, =0,1) and $\phi_t(\mathbf{x})$ (field stored in the automaton, 2D or 3D)
- Time-discrete, local update rules for both fields
- Idea: As in gravitation/electrostatics, the field $\phi_t(\mathbf{x})$ mediates Coulombic attractions between anyons

Poisson's law

$$\nabla^2 \Phi(\mathbf{x}) = \sum_{j=1}^{D} \frac{d^2 \Phi(\mathbf{x})}{dx_j^2} = q(\mathbf{x}) \quad \Phi(\mathbf{x}) = \begin{cases} -\log r & \text{for } D = 2, \\ r^{2-D} & \text{otherwise,} \end{cases}$$

$\Rightarrow$ Goal: approximate via cellular automaton

Discretized Laplacian: $$\nabla^2 \phi(\mathbf{x}) = -2D\phi(\mathbf{x}) + \sum_{\langle \mathbf{y}, \mathbf{x} \rangle} \phi(\mathbf{y})$$

Update rules for fields

$\Rightarrow$Specify dynamical equations (automata update rules) whose stationary solutions satisfy the discretized Poisson equation

$$\phi_{t+1}(\mathbf{x}) = (1-\eta)\phi_t(\mathbf{x}) + \frac{\eta}{2D} \sum_{\langle \mathbf{y},\mathbf{x} \rangle} \phi_t(\mathbf{y}) + q_E(\mathbf{x})$$

$0 < \eta < 1/2$ is a smoothing parameter

Convergence

For fixed anyonic charges q_E, the field converges exponentially towards stationary solutions of the Poisson equation:

$$\|\tilde{\phi}_t - \tilde{\phi}_\infty\|_2 \leq e^{-(\eta\pi^2/D)t/L^2} \|\tilde{\phi}_0 - \tilde{\phi}_\infty\|_2$$

- Tilde: Field is rescaled such that overall sum vanishes (irrelevant, since only gradient matters)
- Convergence takes L^2 time: diffusive spreading of information
- However: spread over log(L) distance (maximal error cluster size) suffices

Update rules for anyons

- After *c* rounds of of updates of the field $\phi_t(\mathbf{x})$ perform one update of the anyon locations (*c* is the "field velocity")
- Each anyon moves with probability ½ to the (unique) neighboring cell with largest field value
- Repeat until there are no anyons left

Pseudocode of algorithm

Repeat c ***times:***

  Parallel for all $\boldsymbol{\xi} \in \Lambda$***:***

    $$\phi(\boldsymbol{\xi}) = \operatorname*{avg}_{\langle \boldsymbol{\xi}', \boldsymbol{\xi} \rangle} \phi(\boldsymbol{\xi}') + q(\boldsymbol{\xi})$$

Parallel for all $\mathbf{x} \in \mathbb{V}$***:***

  If $q(\mathbf{x}) \neq 0$***:***

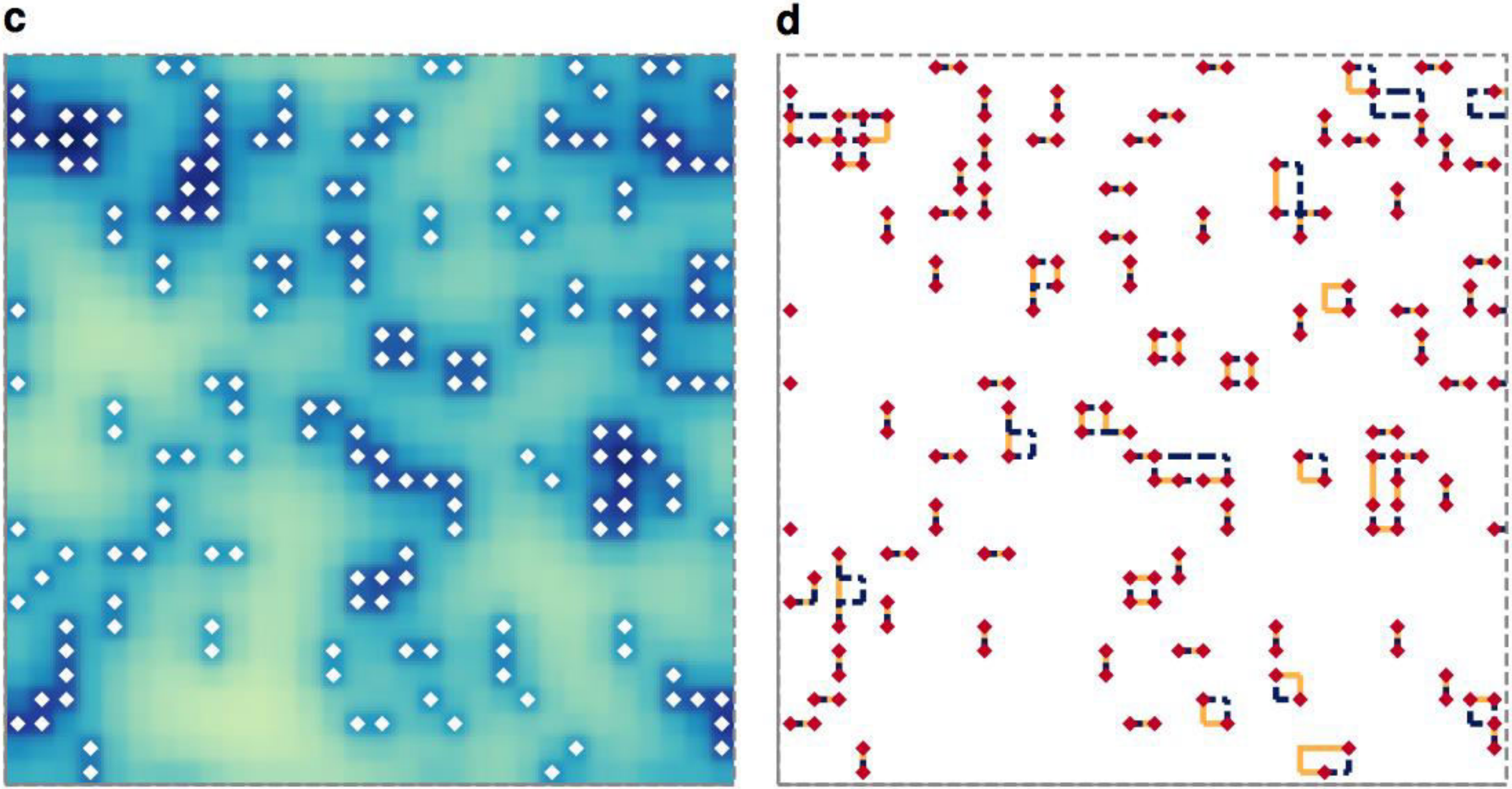
    With probability 0.5***:***

      Move anyon from $\mathbf{x}$ ***to*** $\arg\max_{\langle \mathbf{y}, \mathbf{x} \rangle} \phi(\mathbf{y})$

If still anyons present:

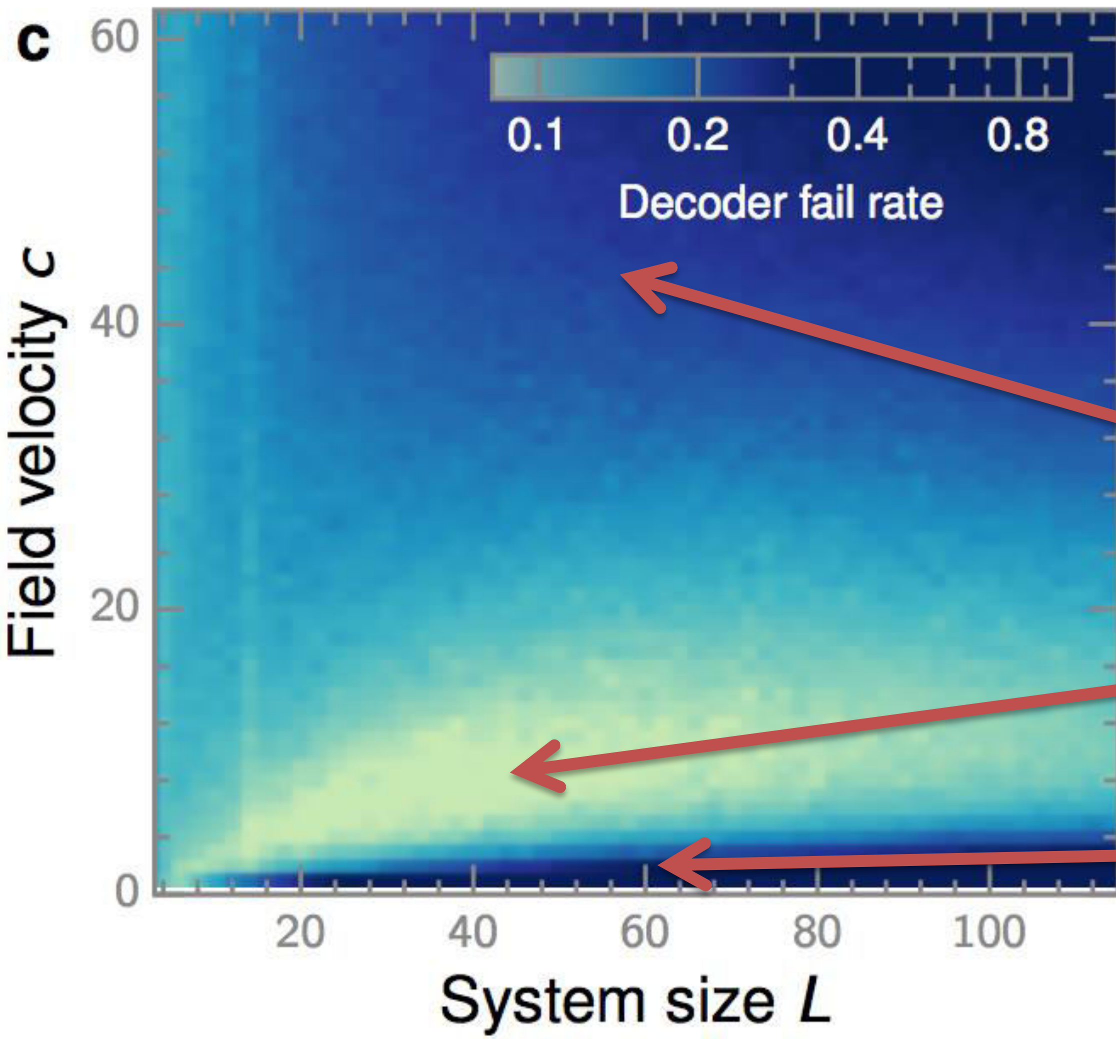
  $\tau = \tau + 1$

  Go to beginning



d) red: anyons; orange: actual errors; blue: recovery paths dictated by the decoder

Result for 2D field (p=6%)



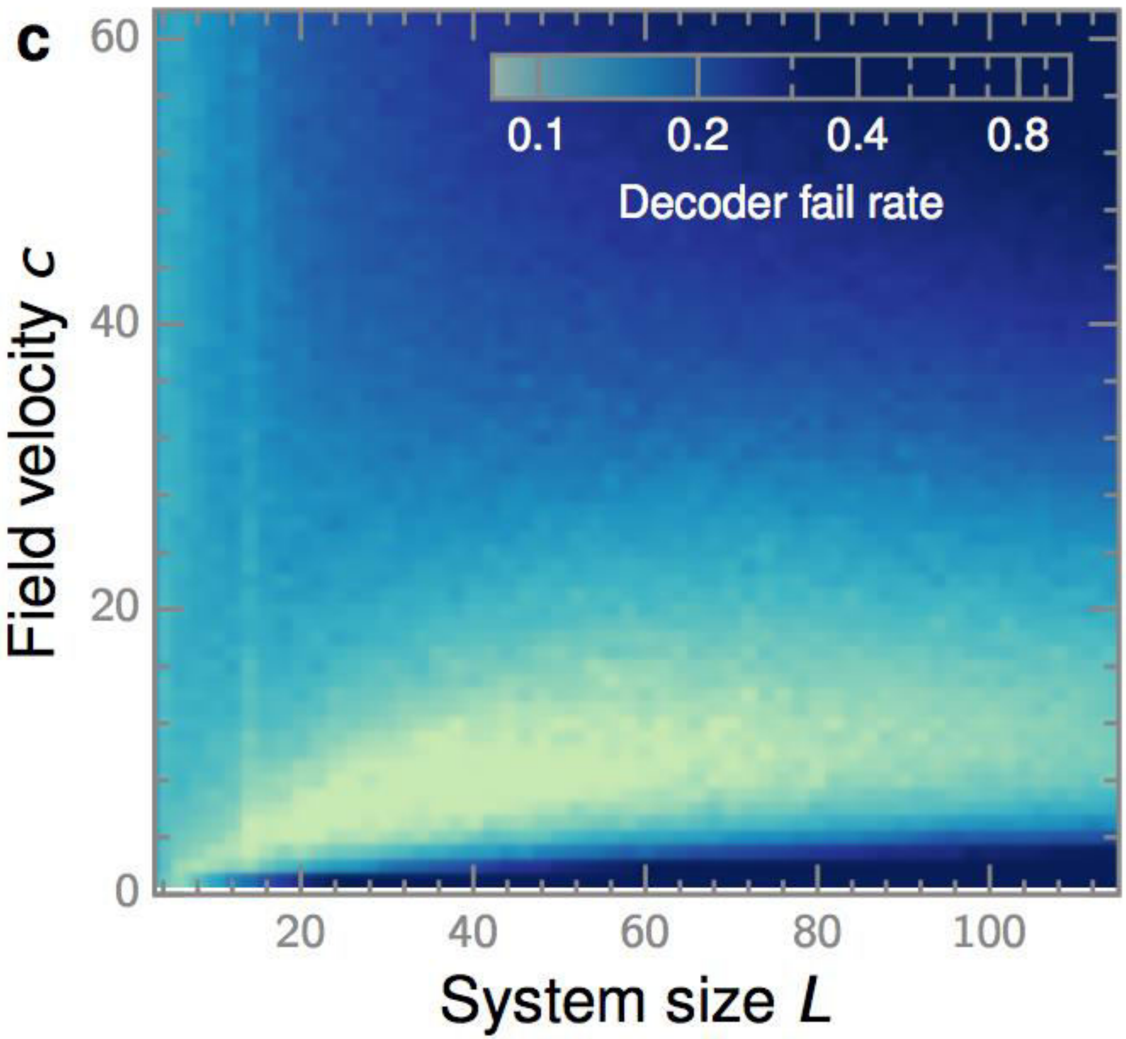
No threshold found. Failure rate > p for all parameters!

Field too long-ranged

"Sweet spot"

Self-interaction too strong (see later)

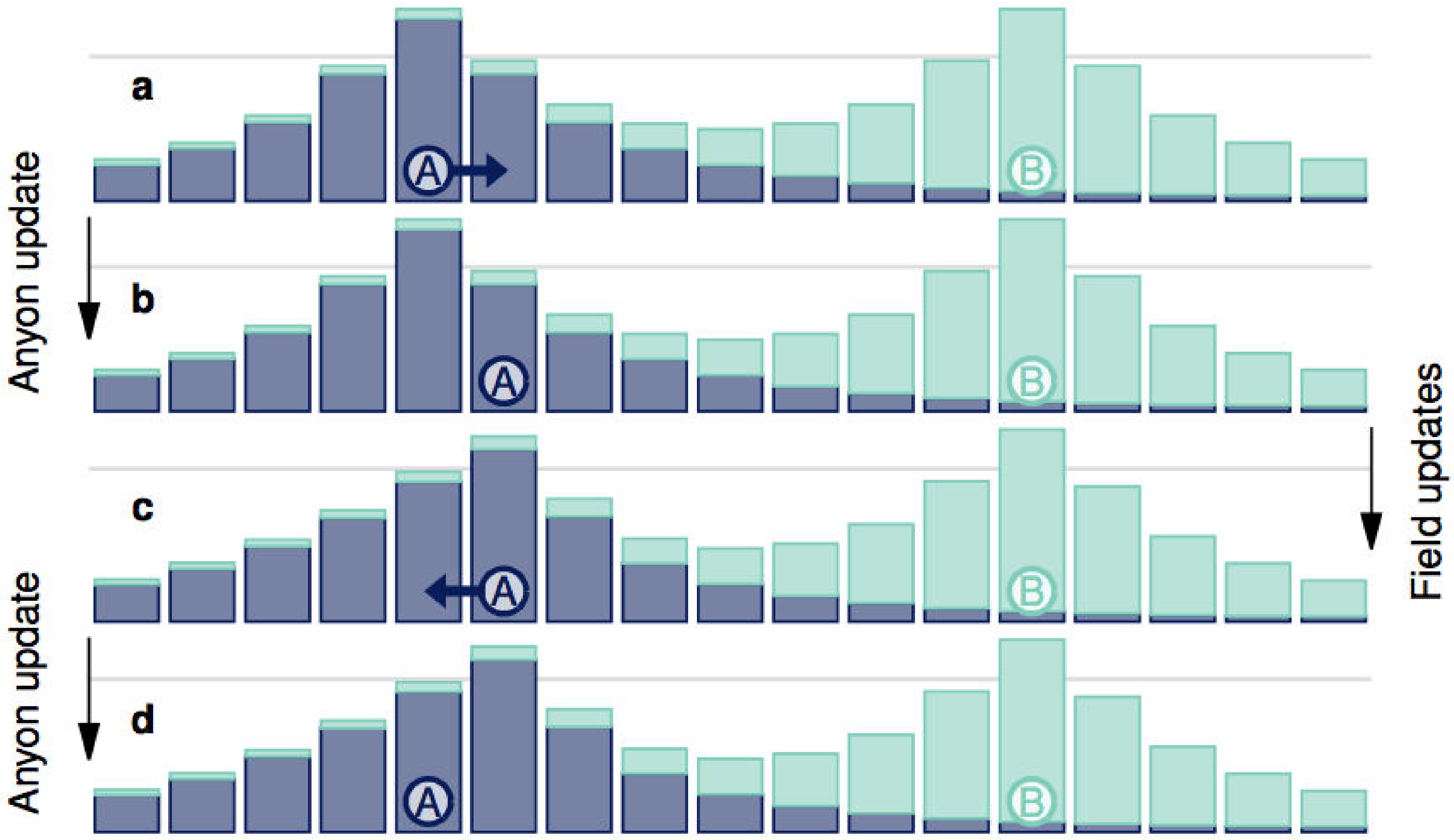
Result for 2D field (p=6%)



No threshold found
Failure rate > p for all parameters!

“While the 2D decoder is expected to be most suitable for modest storage needs, no strict asymptotic threshold exists” ??

Self-interaction



The field velocity can't be constant

- Below percolation, typical clusters of errors/anyons are of size log(L)
- => The field needs to propagate a distance log(L) for the decoder to converge in a time which is sub-exponential in L
- => Lower bound on the field velocity: $c \geq c_{\min} \sim \log^2(L)$

"If *c* were to be taken independent of L, then beyond a critical cluster size, the field contribution due to an anyon's self-interaction would dominate the contribution from the anyons at the other end of the cluster, hence causing persistent flickering of the anyons and preventing the decoder from converging in a reasonable time."

Further difficulties in 2D

- We want update rules to be time-independent
- Seems not possible in 2D

- “From numerical simulations, we are lead to conclude that the 2D equilibrium field is too long range, and tends to break the cluster structure of the anyons by extending error strings rather than shrinking them. In higher dimensions the field profile decays steeply enough so that this is not the case.”

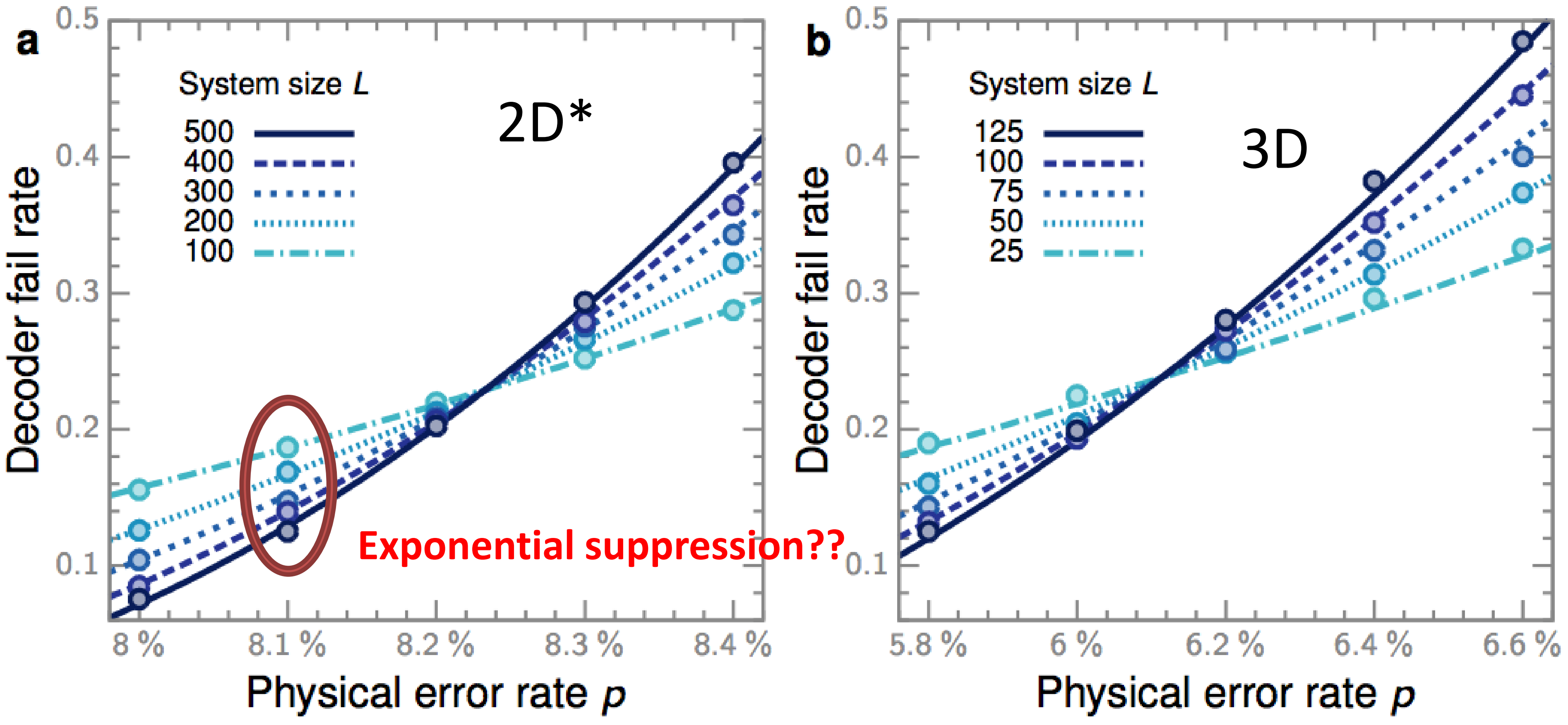
- (Generalizability to continuous error correction?)

Improved field dynamics

	2D	2D*	3D
Lattice (Λ)	$L \times L$	$L \times L$	$L \times L \times L$
Field velocity (c)	τ indep.	$1 + 0.2 \cdot \tau$	$10 \cdot \log^2(L)$
Threshold (p_{th})	N/A	$8.2\,\%$	$6.1\,\%$
Required sequences (τ_{RT})	N/A	$o(\log^{2.5}(L))$	$o(\log(L))$

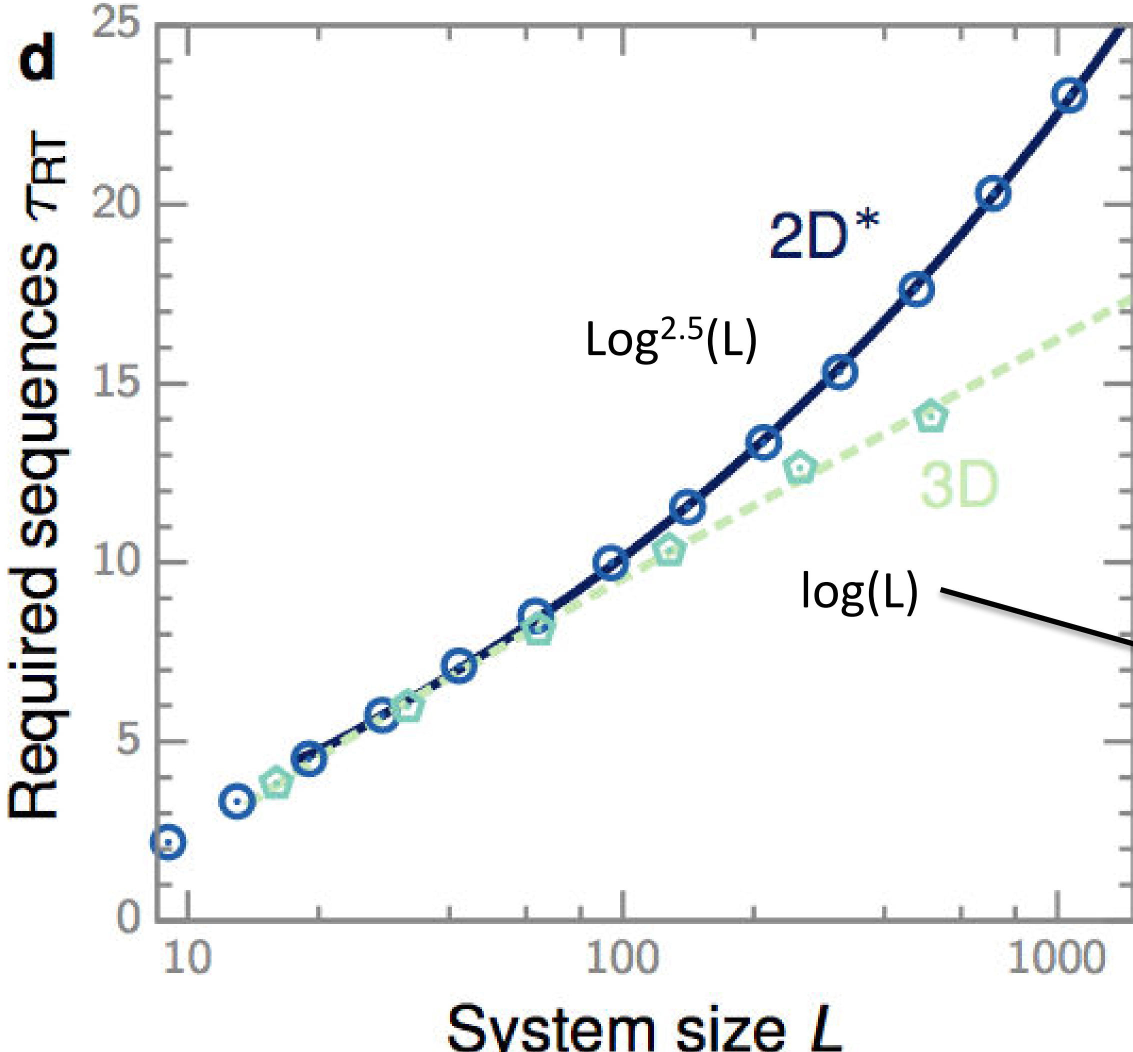
τ refers to the sequence index, where each sequence contains $c + 1$ elementary updates per cell. The decoder terminates after τ_{RT} sequences on average. For the 2D* decoder the rules for the field updates can be thought of as time dependent since c increases with τ. The smoothing parameter as defined in Eq. (4) is $\eta = 1/2$.

Thresholds for improved automata



Again: Decoder fail rate is > p for all data points presented

Required number of iterations



p=1%

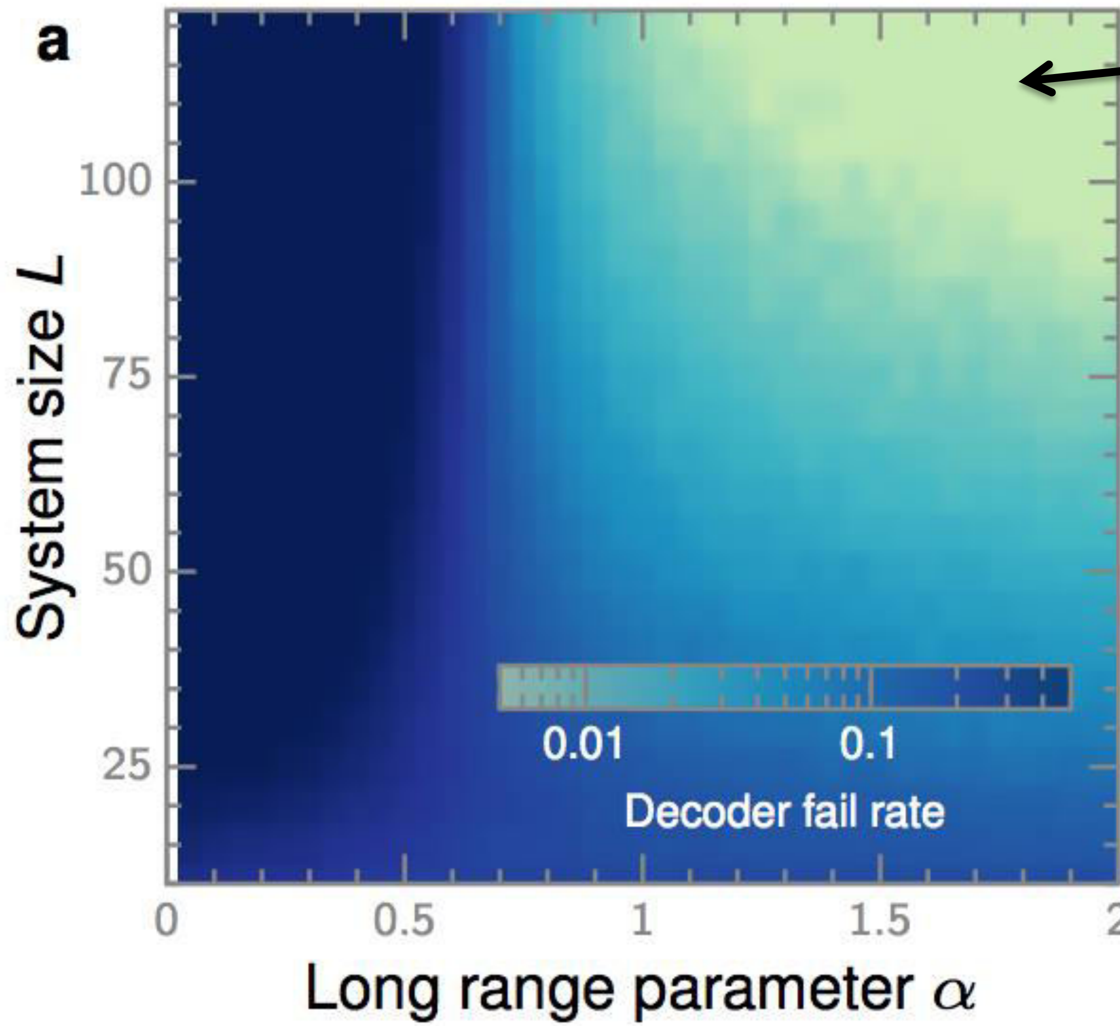
10 $\log^2(L)$ updates each

Large-c regime

- Investigate the case where the fields are always given by their stationary values (*not* a cellular automaton) with the field of an anyon at the origin given by $\Phi(r) = r^{-\alpha}$

- The gradient is proportional to $r^{-(\alpha+1)}$, so for large L the gradient produced by all other anyons in the code is only finite for $\alpha > 1$

Large-c regime (p=5%)

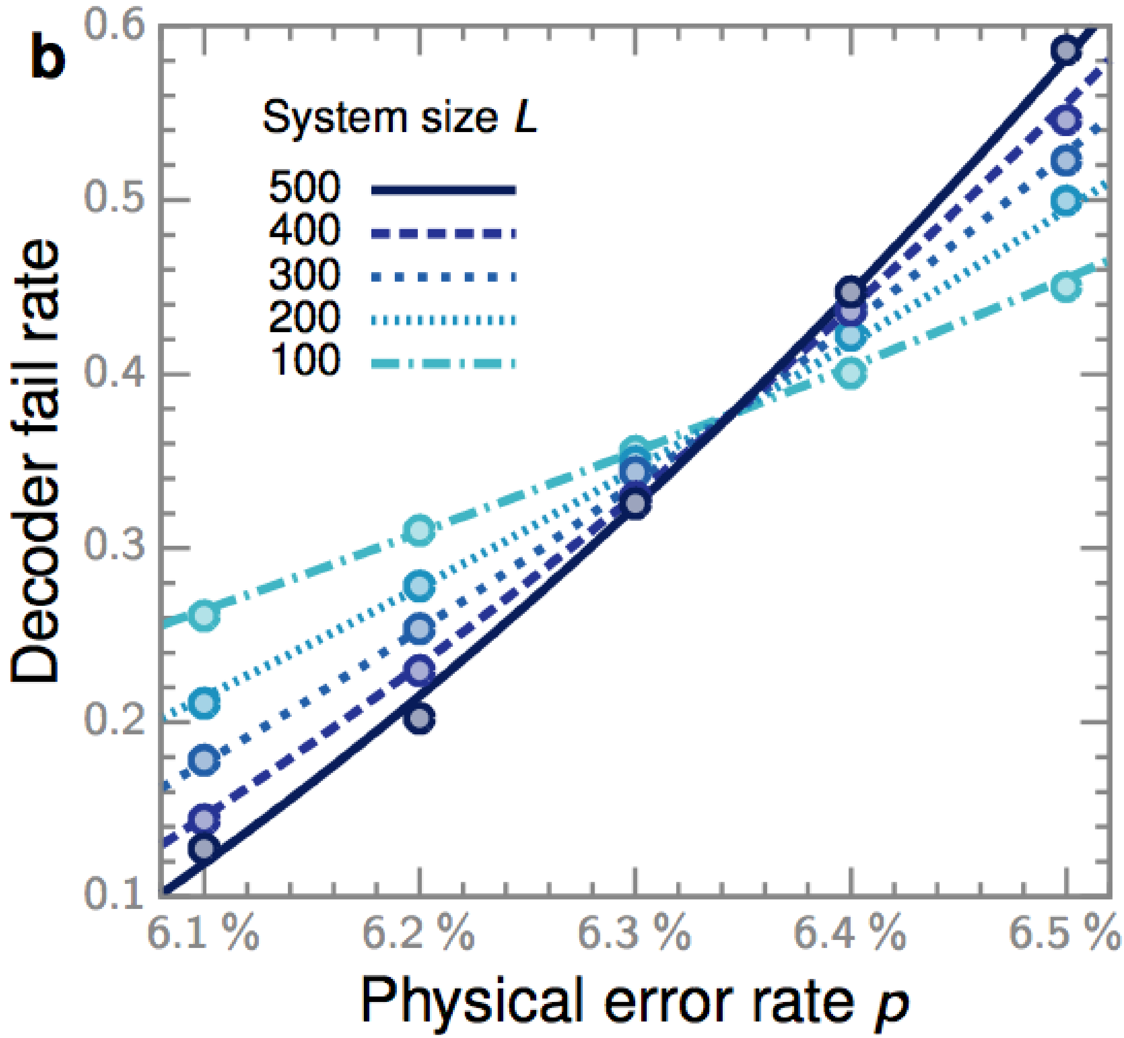


< p ☺

Note: $\alpha = 1$ corresponds to the harmonic field produced by the 3D automaton

"We highlight that very high values of α are not favorable in general, since they require increased precision in the field resolution."

Failure rates for α=1



Very similar to 3D automaton $\Rightarrow$ c = 10 $\log^2(L)$ sufficient for good equilibration of the field

Conclusions

- They have introduced a new class of decoders: pair anyonic excitations by mediating long range information through an auxiliary field.
- The decoder has an intrinsically parallelized architecture.
- A threshold is obtained for a 3D decoder with time-homogeneous update rules and for a 2D decoder with time-dependent update rules.
- *Note by AH:* Whether the standard decoder (MWPM) can be performed in a strictly local way or requires O(log L) communication length is under dispute.
- Main drawbacks: field velocity and required numerical resolution scale (poly-)logarithmically with L. O(1) requirements seem in principle achievable.
- To Do: correlated noise, syndrome measurement errors.