Supersymmetry in the Majorana Cooper-Pair Box

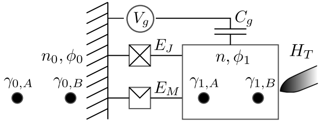
Jascha Ulrich, İnanç Adagideli, Dirk Schuricht, and Fabian Hassler

Over the years, supersymmetric quantum mechanics has evolved from a toy model of high energy physics to a field of its own. Although various examples of supersymmetric quantum mechanics have been found, systems that have a natural realization are scarce. Here, we show that the extension of the conventional Cooper-pair box by a 4π-periodic Majorana-Josephson coupling realizes supersymmetry for certain values of the ratio between the conventional Josephson and the Majorana-Josephson coupling strength. The supersymmetry we find is a "hidden" minimally bosonized supersymmetry that provides a non-trivial generalization of the supersymmetry of the free particle and relies crucially on the presence of an anomalous Josephson junction in the system. We show that the resulting degeneracy of the energy levels can be probed directly in a tunneling experiment and discuss the various transport signatures. An observation of the predicted level degeneracy would provide clear evidence for the presence of the anomalous Josephson coupling.

Phys. Rev. B **90**, 075408 (2014)

Layout

Majorana Cooper pair box



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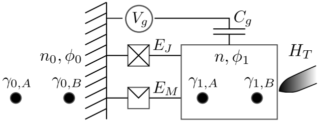

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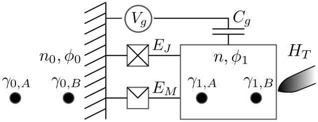

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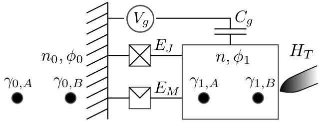

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- large grounded superconductor with zero charging energy, thus no dynamics of ϕ_0 (set to zero)

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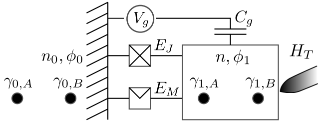

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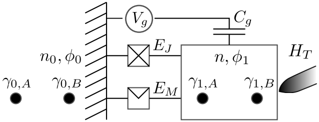

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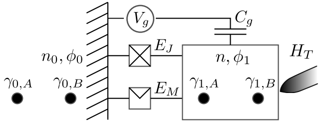

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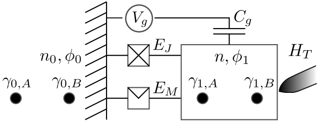

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- $\cos(\pi n_k)\sigma_k^z - \sin(\pi n_k)\sigma_k^y = (-1)^{n_k}$

Supersymmetry (SUSY) in field theories of elementary particles results in a fermion partner to every boson, and vice versa. Both partners have the same mass (synonymously, energy). The resulting pattern has one non-degenerate, zero-energy, bosonic state (the vacuum), while every other state of the spectrum is degenerate in pairs (a boson and a fermion). This is termed a SUSY spectrum. Witten extended the SUSY concept to ordinary quantum mechanics.

E. Witten, Nucl. Phys. B **188**, 513 (1981)

Definition: The quantum mechanical system $(\mathcal{H}, H)$ is said to be supersymmetric if there exists a finite number of self-adjoint operators $Q_1, \ldots, Q_n$ (called supercharges) as well as a self-adjoint operator K (called involution), all of which operators act on $\mathcal{H}$ and satisfy $K^2 = 1$, $\{Q_i, K\} = 0$ and $\{Q_i, Q_j\} = 2\delta_{ij}H$

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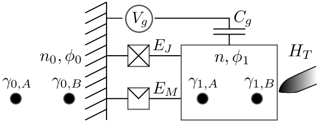

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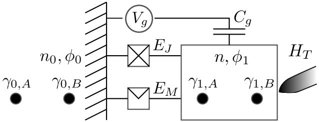

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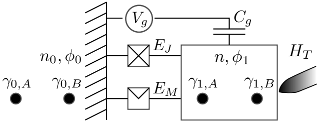

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- $H = E_C(n - n_g)^2 + E_J\big(1 - \cos\phi\big) + E_M \cos(\phi/2)$
- Showing that H is supersymmetric amounts to finding a supercharge Q and an involution K realizing the SUSY algebra

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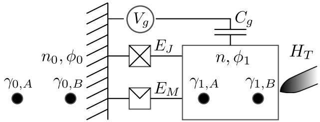

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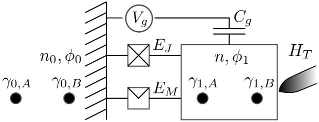

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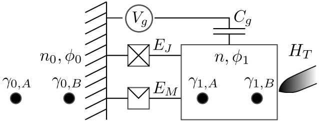

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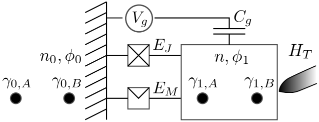

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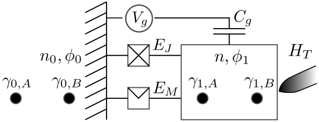

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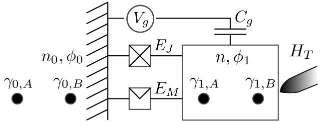

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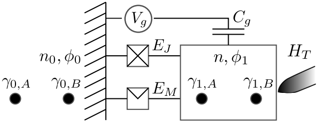

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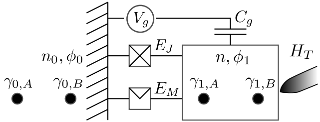

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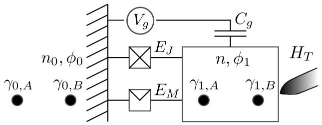

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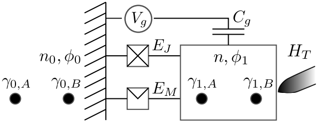

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- Cannot be realized in present system since the Majoranas are no longer an independent degree of freedom due to the fermion parity constraint !

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- In this sense it provides a clear signature of the Majorana-induced 4π-periodic Josephson relation.

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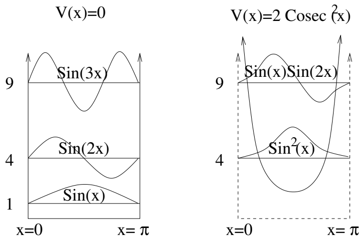

Figure : $\hbar = 2m = 1,\ E_n = 1^2, 2^2, 3^2, \cdots$

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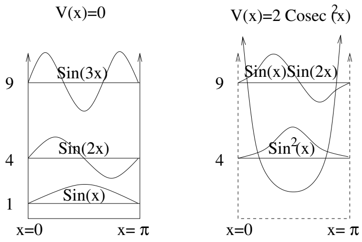

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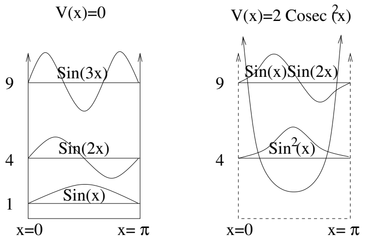

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Particle in a box

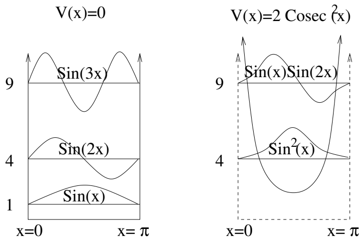

Figure : $\hbar = 2m = 1,\ E_n = 1^2, 2^2, 3^2, \cdots$

- Infinite square well potential (V=0 and L=π) and its supersymmetric partner potential $2\mathrm{cosec}^2(x)$
- Supersymmetric states have opposite parity
- Unique ground state with even parity

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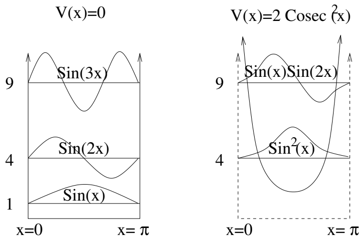

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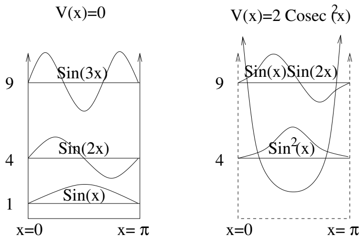

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- Discrete SUSY spectra evolves as $L \to \infty$

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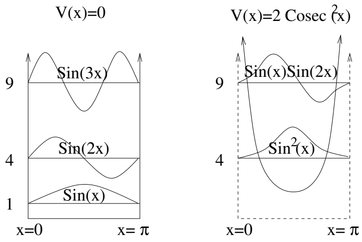

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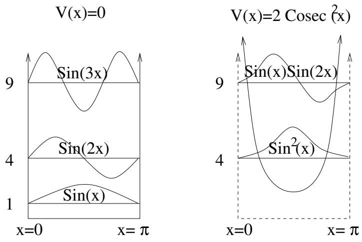

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- Discrete SUSY spectra evolves as $L \to \infty$
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- Continuum of even and odd parity modes but unique even parity ground state

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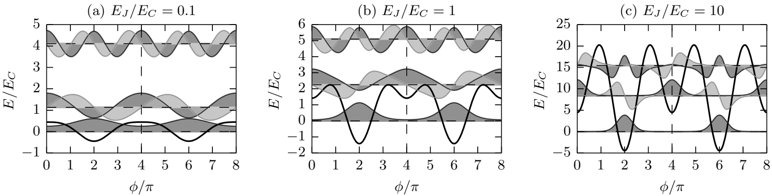
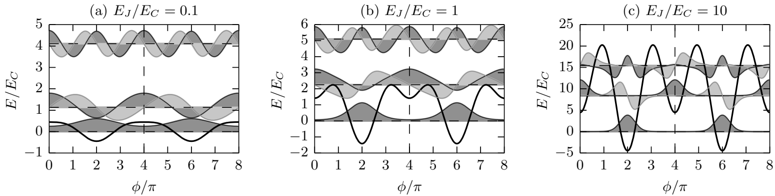


Figure : Numerically calculated wave functions $\psi_{n,\pm}(\phi)$ of the Hamiltonian for $n = 0, 1, 2$ at the supersymmetric point $n_g = 0$, $E_M = \sqrt{2E_J E_C}$ for different values of E_J/E_C (and thus $\alpha = \sqrt{2E_J/E_C}$). The wave functions are chosen real and are aligned at their corresponding eigenenergies. States $\psi_{n,+}(\phi)$ in the even parity sector of the superconducting phase are plotted in dark gray while the states $\psi_{n,-}(\phi)$ in the odd parity sector are plotted in light gray. The black line represents the underlying potential. Both the potential and the states are 4π periodic. We note that due to the supersymmetry, all levels with $n > 0$ are doubly degenerate.



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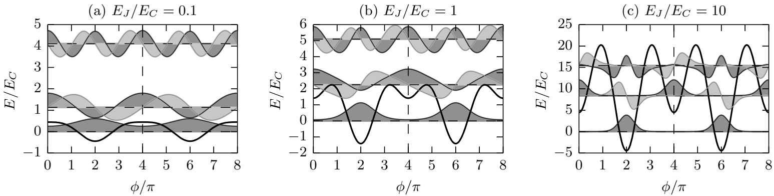
- $V = E_M \cos(\phi/2) + E_J(1 - \cos\phi)$



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- $V = E_M \cos(\phi/2) + E_J(1 - \cos\phi)$
- For $\alpha \to \infty$ the states are well localized close to the minima at $\phi \in 2\pi\mathbb{Z}$

Hidden Supersymmetry in Majorana Cooper pair box

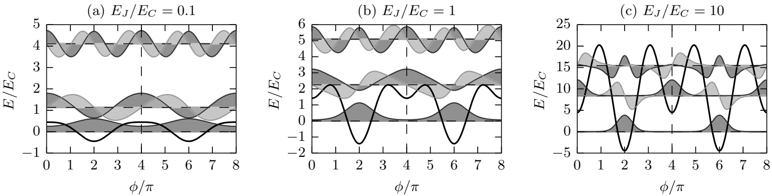


Figure : Numerically calculated wave functions

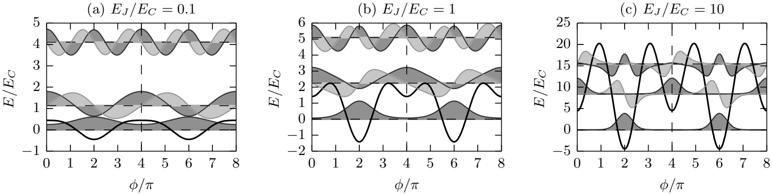
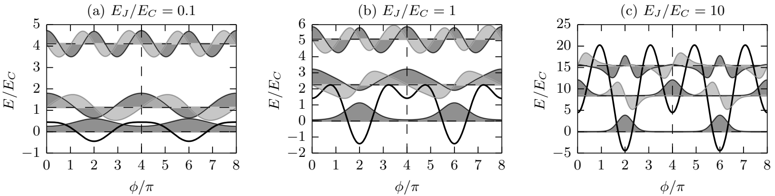


Figure : Numerically calculated wave functions

- Close to $\phi = 2\pi$, we have $V_{2\pi} \approx -E_M + \frac{1}{2}E_J(\phi - 2\pi)^2$ with the spectrum $E_{2\pi,n} = -E_M + \sqrt{8E_C E_J}(n + \frac{1}{2})$.



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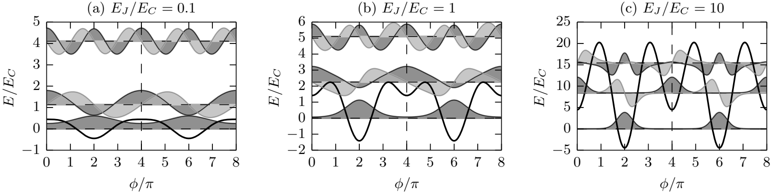
- In the second mimimum, at ϕ close to zero, we have $V_0 \approx E_M + \frac{1}{2}E_J\phi^2$ which leads to the approximate spectrum $E_{0,n} = E_M + \sqrt{8E_C E_J}(n + \frac{1}{2})$



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- At the supersymmetric point ($E_M = \sqrt{2E_J E_C}$, $n_g = 0$), we observe the degeneracy $E_{2\pi,n+1} = E_{0,n}$ valid for $n \geq 0$.

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- The degeneracies of the whole spectrum (except for the ground state) arising in this model at the supersymmetric point can be directly probed by a tunneling experiment.