Absence of Quantum Time Crystals in Ground States

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In analogy with crystalline solids around us, Wilczek recently proposed the idea of "time crystals" as phases that spontaneously break the continuous time translation into a discrete subgroup. The proposal stimulated further studies and vigorous debates whether it can be realized in a physical system. However, a precise definition of the time crystal is needed to resolve the issue. **Here we first present a definition of time crystals based on the time-dependent correlation functions of the order parameter. We then prove a no-go theorem that rules out the possibility of time crystals defined as such, in the ground state of a general Hamiltonian which consists of only short-range interactions.**

arxiv:1410.2143

Wilczek's proposal

Particle on a ring, threaded by a flux:

Lagrangian: $L = \frac{1}{2}\dot{\phi}^2 + \alpha\dot{\phi}$

Hamiltonian: $H = \frac{1}{2}\left(\pi_\phi - \alpha\right)^2$

Eigenstates: $|\ell\rangle = e^{i\ell\phi}$ (ℓ integer)

$\langle\ell|H|\ell\rangle = \frac{1}{2}(\ell - \alpha)^2$

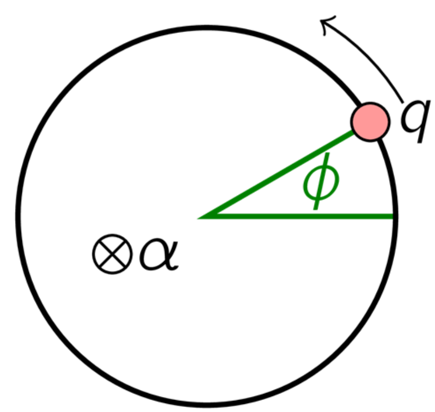
Symmetry and its spontaneous breaking is a central theme in modern physics. Perhaps no symmetry is more fundamental than time-translation symmetry, since time-translation symmetry underlies both the reproducibility of experience and, within the standard dynamical frameworks, the conservation of energy. So it is natural to consider the question, whether time-translation symmetry might be spontaneously broken in a closed quantum-mechanical system. That is the question we will consider, and answer affirmatively, here.

F. Wilczek, PRL **109**, 160401 (2012)

Ground state: $\langle\ell|\dot{\phi}|\ell\rangle = \ell - \alpha \neq 0$ (for non-integer α)

Contradiction to Heisenberg equation?

$\pi_\phi = \dot{\phi} + \alpha$



No time-dependent observables:

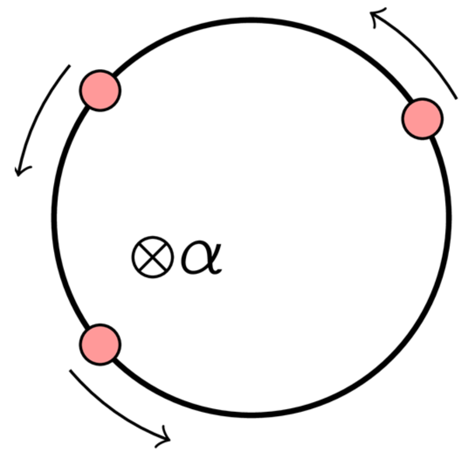
- The particle is delocalized
- The current is constant over time
- **How to get localized particles?**

discussed in Fabio's JC talk 05/02/2013

The proposed example

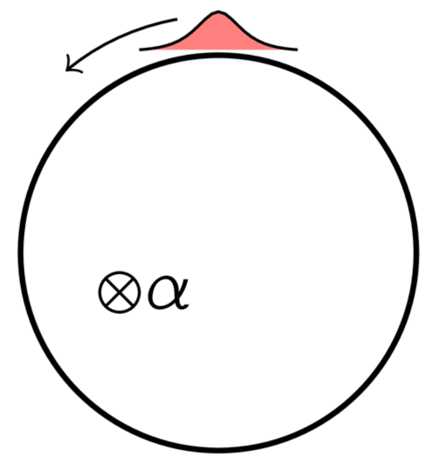
N particles with weak attractive interactions:

$$H = \sum_{j=1}^{N} \frac{1}{2}(\pi_j - \alpha)^2 - \frac{\lambda}{N-1} \sum_{j \neq k} \delta(\phi_j - \phi_k)$$



Wilczek's solution:

- First, $\alpha = 0$
- Mean-field approximation (effective one-body potential)
- Gauge transformation to get solution for nonzero α
- Ground state is a rotating soliton

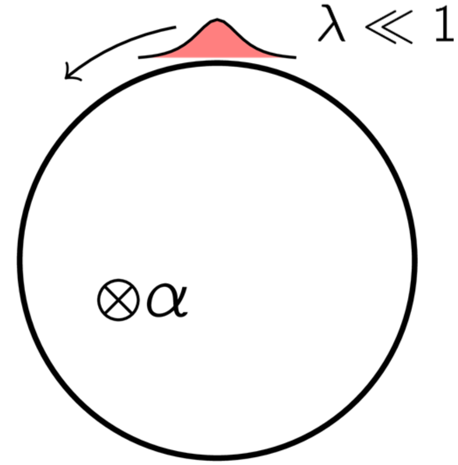


F. Wilczek, PRL **109**, 160401 (2012)

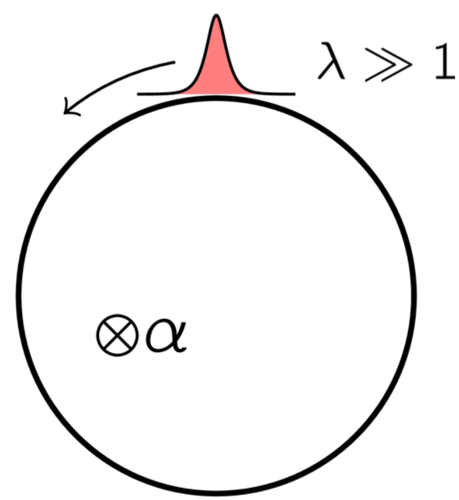
Early criticism

Problems with Wilczek's solution:

- In the limit $\lambda \to \infty$, the soliton should be insensitive to the flux
- The rotating lump should radiate, losing energy



These remarks strongly suggest that Wilczek's rotating soliton is *not* the ground state and that the true ground state is actually a stationary state, as I show below. The solution of the NLSE for arbitrary flux [3] is too lengthy and technical to fit in this Comment; thus, I shall give here



In the light of the above discussion, it seems that the very existence of "quantum time crystals" remains highly speculative.

P. Bruno, PRL **110**, 118901 (2013)

Spontaneous symmetry breaking

Bogoliubov's prescription to find spontaneous symmetry breaking in the thermodynamic limit:

- Add symmetry-breaking perturbation with strength h
- Take the thermodynamic limit $V \to \infty$, $N/V = \text{const.}$
- Take the limit $h \to 0$

$$\lim_{h\to 0}\lim_{V\to\infty} \langle 0|\rho(x)|0\rangle$$

Works great for **AFM, FM, crystalline** order, etc.

Caveats:

- Special Hamiltonian
- Rotating frame with time-independent H exists.

For time crystals:

Response to a time-dependent perturbation:

$$H = \sum_{i=1}^{N}\left[\frac{(\ell_i - \alpha_i)^2}{2m_i R^2} + h_i(\phi_i - \Omega t)\right] + \sum_{j\neq k} U_{jk}(\phi_j - \phi_k)$$

Energy per particle:

$$\lim_{h\to 0}\lim_{V\to\infty} \frac{1}{N}\left(E_{\alpha,0}^{(\Omega)} - E_{\alpha,0}^{(0)}\right) > 0$$

Non-rotating ground state always has the smallest energy. Only for excited states, the minimum can have nonzero Ω.

P. Bruno, PRL **111**, 070402 (2013)

Definition in terms of long-range order

Define spontaneous symmetry breaking using long-range order of correlation functions:

$$\sigma^2 := \lim_{|x-x'|\to\infty} \lim_{V\to\infty} \langle 0|\phi(x,0)\phi(x',0)|0\rangle$$

Example: Crystalline order (periodic in space)

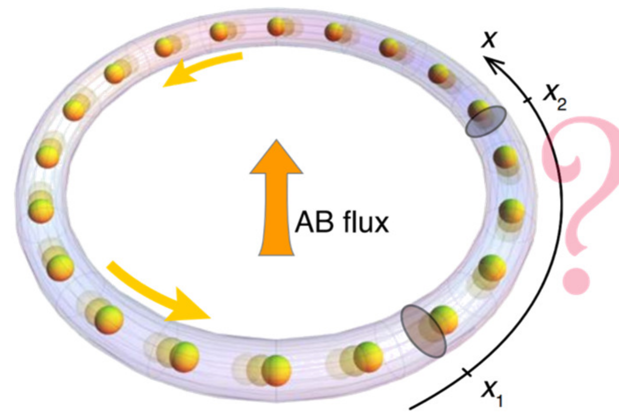
$$\lim_{|x-x'|\to\infty} \lim_{V\to\infty} \langle 0|\rho(x,0)\rho(x',0)|0\rangle = f(x-x')$$

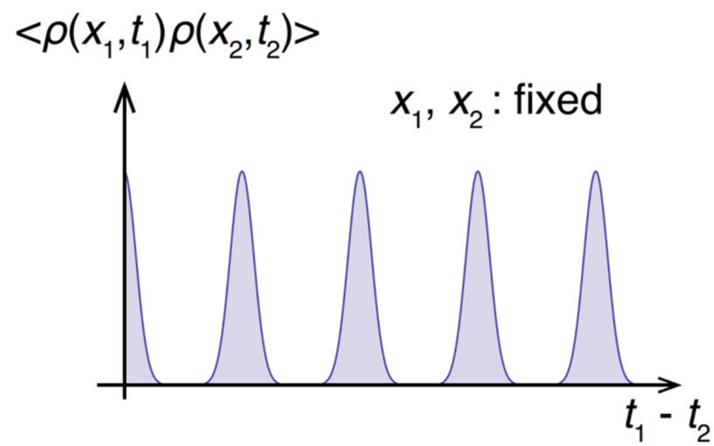
By analogy, define time crystals as

$$\lim_{|x-x'|\to\infty} \lim_{V\to\infty} \langle 0|\phi(x,t)\phi(x',t')|0\rangle = f(t-t')$$

Why long-range spatial order?

- ➢ Otherwise too many "trivial" time crystals exist.





Watanabe and Oshikawa, arxiv:1410.2143

Does (such) a time crystal exist?

Assumptions:
- Locality of the Hamiltonian
- Locality of commutator

$$H = \int d^d x \mathcal{H}(x)$$

$$[\phi(x,t), \phi(x',t)] \approx \delta(x - x')$$

$$\lim_{|x-x'|\to\infty} \lim_{V\to\infty} \langle 0|\phi(x,t)\phi(x',t')|0\rangle = f(t-t')$$

$$\lim_{V\to\infty} \frac{1}{V^2} \langle 0|\Phi(t)\Phi(t')|0\rangle = f(t-t')$$

Integrated order parameter:

$$\Phi(t) = \int d^d x \phi(x,t)$$

Step 1: A time crystal has long-range order

$$(*) \quad \lim_{V\to\infty} \int_0^\infty d\Omega \rho(\Omega) e^{-i\Omega t} = f(t) \qquad \rho(\Omega) = \frac{1}{V^2} \sum_n |\langle 0|\Phi(0)|n\rangle|^2 \delta(\Omega - E_n) \geq 0$$

$$\Longrightarrow \quad 0 \leq |f(t)| \leq f(0)$$

$\Longrightarrow$ Spatial long-range order because $f(0) > 0$

Does (such) a time crystal exist?

$$(*)\quad \lim_{V\to\infty}\int_0^\infty d\Omega\rho(\Omega)e^{-i\Omega t} = f(t)$$

Step 2: Investigate the state $|\Phi\rangle = \frac{\Phi|0\rangle}{|\Phi|0\rangle|}$

$$\langle\Phi|H|\Phi\rangle = \frac{\langle 0|[[\Phi(0), H], \Phi(0)]|0\rangle}{2\langle 0|\Phi(0)^2|0\rangle} \propto \frac{V^3V^{-2}}{f(0)V^2} = O(V^{-1})$$

$$(**)\quad \int_0^\infty d\Omega\rho(\Omega)\Omega = O(V^{-1}) \qquad \rho(\Omega) = \frac{1}{V^2}\sum_n |\langle 0|\Phi(0)|n\rangle|^2\delta(\Omega - E_n) \geq 0$$

The results (*) and (**) are mutually contradictory. Therefore, $f(t) \equiv 0$, so no time crystals exist (at least according to the authors' definition).

- Small $O(V^{-1})$ order parameter can exist in finite systems
- The proof remains true also for space-time crystals
- It does not hold for excited states or nonequilibrium states

Conclusions

- Conventional concepts of spontaneous symmetry breaking are hard to generalize to time crystals.
- The authors' definition is based on the long-distance asymptotic behavior of correlation functions.
- According to this definition, local Hamiltonians cannot break time translation invariance spontaneously.

H. Watanabe and M. Oshikawa, arxiv:1410.2143