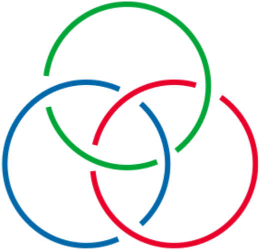
Braiding statistics of loop excitations in 3D
by Chenjie Wang and Michael Levin (PRL 113, 080403 (2014))



Constantin Schrade

University of Basel

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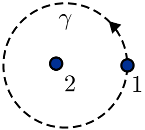
Braiding statistics in 2D
$=$ Part of θ_{12}, U_{12} that only depends on the topology of the path γ



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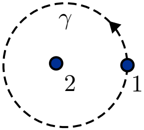

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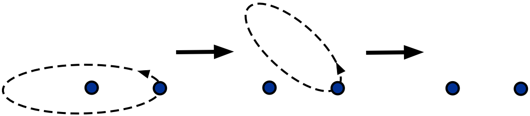
Braiding statistics in 3D

Point-like particles have trivial braiding statistics in 3D!



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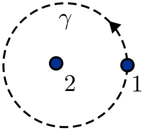

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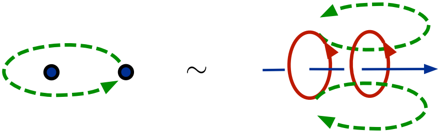
Braiding statistics in 3D

3D systems can have loop excitations with non-trivial braiding!



Main result

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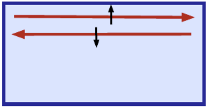

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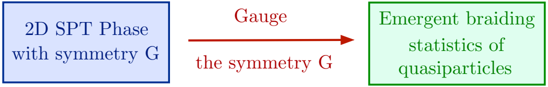

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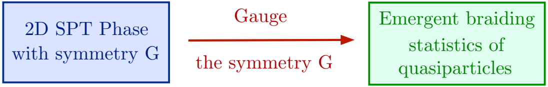

Different 2D SPT phases with the same symmetry G correspond to different emergent braiding statistics of quasiparticles.

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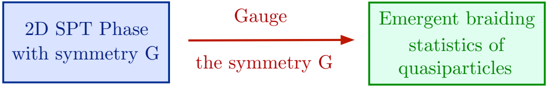

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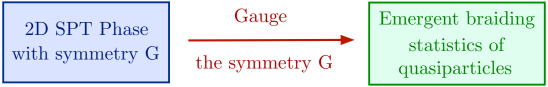

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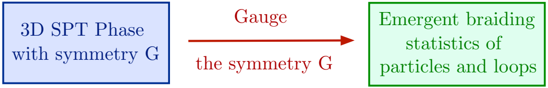

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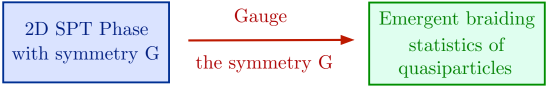

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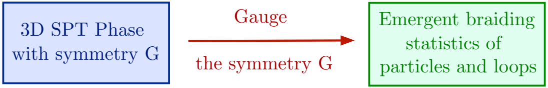


Different 3D SPT phases with the same symmetry G correspond to different emergent three-loop braiding statistics.

Outline

Emergent braiding statistics in 2D SPT Order

- $\mathbb{Z}_2$ Symmetry Protected Topological Order
- Gauging the $\mathbb{Z}_2$ symmetry
- Emergent braiding statistics of quasiparticle excitations

Emergent braiding statistics in 3D SPT Order

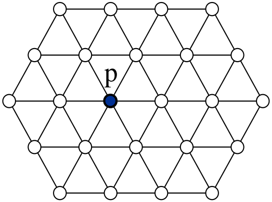
- Extending the results to 3D SPT phases
- Braiding statitics of particles and loops
- Three loop braiding statistics

Question I

How do we physically characterize different 2D SPT phases?

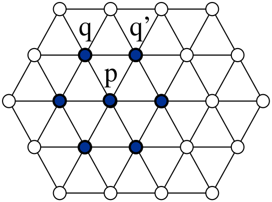
$\mathbb{Z}_2$ Symmetry Protected Topological Order

"Trivial" Paramagnet



$$H_0 = -\sum_p \sigma_p^x$$

"Topological" Paramagnet



$$H_1 = -\sum_p B_p$$

$$B_p = -\sigma_p^x \cdot \prod_{\langle pqq' \rangle} i^{\frac{1-\sigma_q^z \sigma_{q'}^z}{2}}$$

$$\mathbb{Z}_2 - \text{Symmetry} = \prod_p \sigma_p^x = \text{Global Spin Flip}$$

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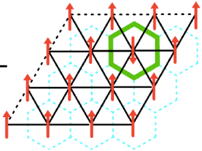
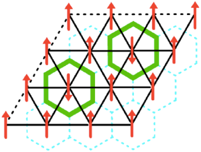
$$|\Psi_0\rangle \propto \frac{1}{2}\prod_p (1+\sigma_p^x)\,|\text{All}\uparrow\rangle \propto (1 + \sum_p \sigma_p^x + \sum_{p,p'} \sigma_p^x \sigma_{p'}^x + \ldots)\,|\text{All}\uparrow\rangle$$

$\propto$ +  +  + ...

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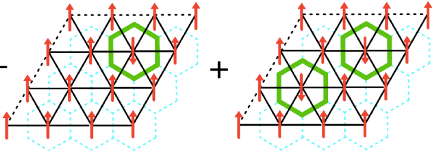
$\propto$ +  + ...

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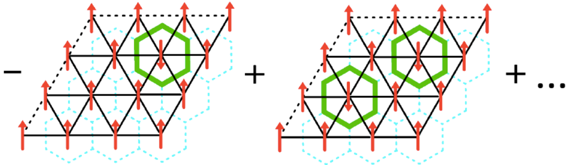
$\propto$ 

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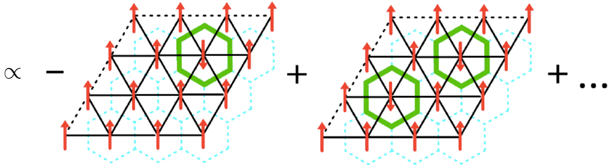

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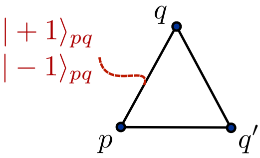

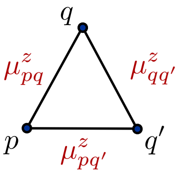

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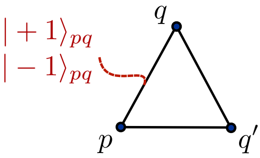

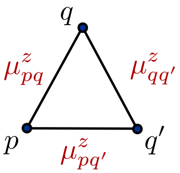

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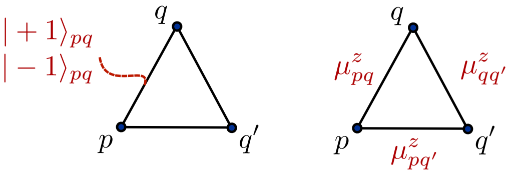

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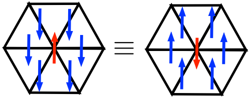

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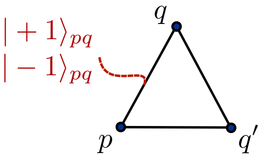

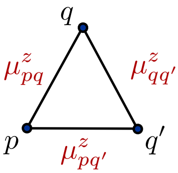

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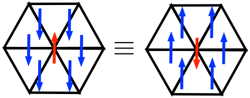

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$\Pi_p = \frac{1}{2} \prod_{\langle pqr \rangle} (1 + \mu_{pq}^z \mu_{qr}^z \mu_{rp}^z)$ projects on states with "zero flux" through the adjoining triangles.

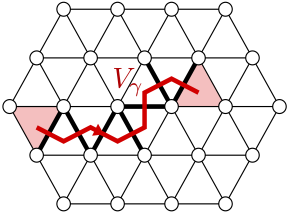
Ensure that "zero flux" states have the lowest energy.

Emergent braiding statistics

$$H_0' = -\sum_p \sigma_p^x \Pi_p - \sum_{\langle pqr \rangle} \mu_{pq}^z \mu_{qr}^z \mu_{rp}^z \quad H_1' = -\sum_p \widetilde{B}_p \Pi_p - \sum_{\langle pqr \rangle} \mu_{pq}^z \mu_{qr}^z \mu_{rp}^z$$

"π-flux" excitation
=Triangle with $\mu_{pq}^z \mu_{qr}^z \mu_{rp}^z = -1$

They are created at the end of strings V_γ



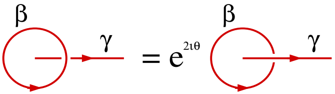
String operator algebra:

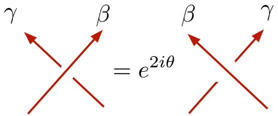
$$V_\beta^0 V_\gamma^0 = V_\gamma^0 V_\beta^0 \qquad \text{(for } H_0'\text{)}$$

$$V_\beta^1 V_\gamma^1 = -V_\gamma^1 V_\beta^1 \qquad \text{(for } H_1'\text{)}$$

Emergent braiding statistics

The string algebra fixes the braiding statistics of π-flux excitations!

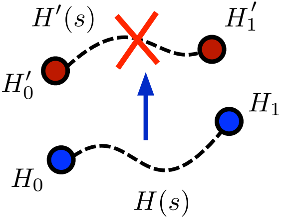




For H_0' : $\theta = 0, \pi \rightarrow$ π-fluxes are bosons or fermions

For H_1' : $\theta = \pm\frac{\pi}{2} \rightarrow$ π-fluxes are "semions"

This result provides a physical distinction between the two paramagnets!



Outline

Question II

How do we physically characterize different 3D SPT phases?

Extending the results to 3D

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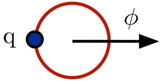
2. Gauge the symmetry.
3. The excitations are

Charges

Gauge charge

$$q = (q_1, ..., q_K) \quad q_i \in \{0, ..., N-1\}$$

Vortex Loops



Gauge flux

$$\phi = (\phi_1, ..., \phi_K) \quad \phi_i \in \frac{2\pi}{N}\ell$$

Gauge charge

$$q = (q_1, ..., q_K) \quad q_i \in \{0, ..., N-1\}$$

Extending the results to 3D

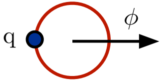
1. Write down a 3D spin-model with symmetry group $(\mathbb{Z}_N)^K$.
2. Gauge the symmetry.
3. The excitations are

Charges

Gauge charge

$$q = (q_1, ..., q_K) \;\; q_i \in \{0, ..., N-1\}$$

Vortex Loops



Gauge flux

$$\phi = (\phi_1, ..., \phi_K) \;\; \phi_i \in \frac{2\pi}{N}\ell$$

Gauge charge

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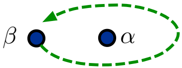
4. Study the braiding statistics of the excitations.

Naive braiding statistics in 3D

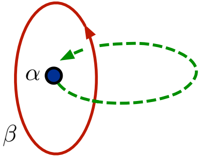
Naive braiding statistics in 3D

Charge-Charge braiding:



$\theta_{\alpha\beta} = 0$

Charge-Loop braiding:



$\theta_{\alpha\beta} = q \cdot \phi$

Naive braiding statistics in 3D

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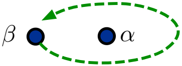

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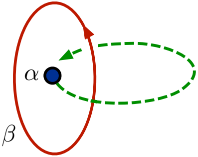
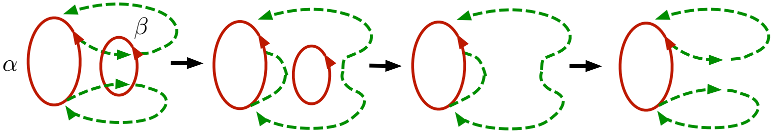


$\theta_{\alpha\beta} = q \cdot \phi$

Neutral Loop-Neutral Loop braiding



$\theta_{\alpha\beta} = 0$

Naive braiding statistics in 3D

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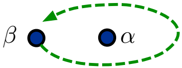

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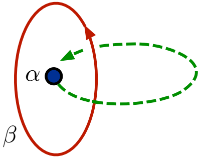

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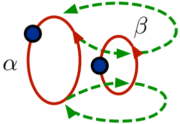
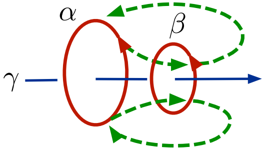


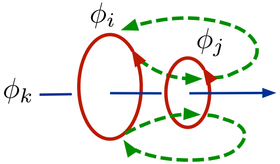
Loop-Loop braiding:

$\theta_{\alpha\beta} = q_\alpha \cdot \phi_\beta + q_\beta \cdot \phi_\alpha$

Three loop braiding statistics in 3D



Three loop braiding statistics in 3D



i-th entry

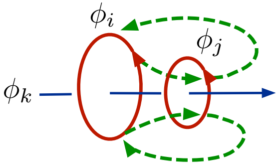
$$\phi_i = \frac{2\pi}{N}(0, ..., 0, 1, 0, ..., 0)$$

Simplifications:

1. Consider only unit flux loops.

   More generic loops are obtained by "fusing" unit flux loops.

Three loop braiding statistics in 3D



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$$\phi_i = \frac{2\pi}{N}(0, ..., 0, 1, 0, ..., 0)$$

Simplifications:

1. Consider only unit flux loops.

   More generic loops are obtained by "fusing" unit flux loops.

2. Consider renormalized Berry phase $\Theta_{ij,k} = N\theta_{ij,k}$

   $\Theta_{ij,k}$ is independent of the charged attached to the loop!

Three loop braiding statistics in 3D

Properties of renormalized Berry phase:

Three loop braiding statistics in 3D

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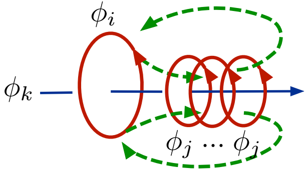

N unit fluxes

1. Total phase $= \mathrm{N}\theta_{ijk}$
2. Total phase $= q \cdot \phi_i = \frac{2\pi}{N} k$ with $k \in \mathbb{Z}$
   Since $N\phi_j = 2\pi\ell \equiv 0$ with $\ell \in \mathbb{Z}$

$\longrightarrow \Theta_{ij,k} = \frac{2\pi}{N} k$ with $k \in \mathbb{Z}$

Example

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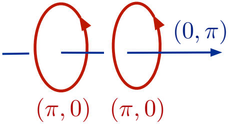
$\Theta_{11,2} =$ 

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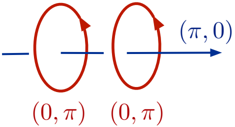
$\Theta_{22,1} =$ 

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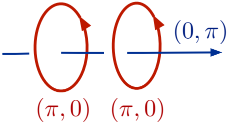
$\Theta_{11,2} =$ 

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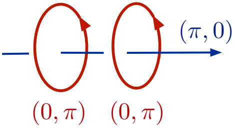
$\Theta_{22,1} =$ 

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This results distinguishes 4 different $\mathbb{Z}_2 \times \mathbb{Z}_2$ 3D SPT phases.

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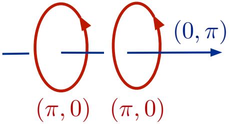
$\Theta_{11,2} =$ 

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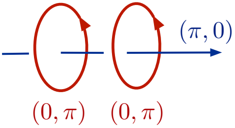
$\Theta_{22,1} =$ 

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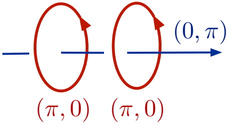
$\Theta_{11,2} =$ 

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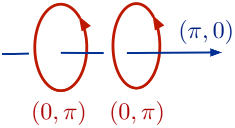
$\Theta_{22,1} =$ 

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More general result:

Different SPT states with the same $(\mathbb{Z}_N)^K$ can be distinguished by three-loop braiding processes.