Causality and quantum criticality with long-range interactions

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Abstract: Quantum lattice systems with long-range interactions often exhibit drastically different behavior than their short-range counterparts. In particular, because they do not satisfy the conditions for the Lieb-Robinson theorem, they need not have an emergent relativistic structure in the form of a light cone. Adopting a field-theoretic approach, we study the one-dimensional transverse-field Ising model and a fermionic model with long-range interactions, explore their critical and near-critical behavior, and characterize their response to local perturbations. We deduce the dynamic critical exponent, up to the two-loop order within the renormalization group theory, which we then use to characterize the emergent causal behavior. We show that beyond a critical value of the power-law exponent of long-range interactions, the dynamics effectively becomes relativistic. Various other critical exponents describing correlations in the ground state, as well as deviations from a linear causal cone, are deduced for a wide range of the power-law exponent.

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Outline

Prerequisites & Systems

Systems considered:

→ Transverse field Ising model (TFIM)

- Standard: $H = -\sum_i \left(\sigma_i^z \sigma_{i+1}^z + g\,\sigma_i^x\right)$.
- Long-ranged: $H_\sigma = -\sum_{i\neq j} \frac{\sigma_i^z \sigma_j^z}{|i-j|^{1+\sigma}} - g\sum_i \sigma_i^x$.

- Non-interacting fermions
  - Standard:
    $H_F = -\sum_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + J\,(c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i) - \mu\, c_i^\dagger c_i$.
  - Long-ranged hopping: $H_F' = H_F + \sum_{i\neq j} \frac{1}{|i-j|^{1+\sigma}}\, c_i^\dagger c_j$.
  - Long-ranged pairing: $H_F' = H_F + \sum_{i>j} \frac{1}{|i-j|^{1+\sigma}}\, c_i^\dagger c_j^\dagger + \mathrm{h.c.}$

Prerequisites from theory

- Lieb-Robinson bounds: Light-cones
- Critical exponents & response function & renormalization group theory

Prerequisites: Lieb-Robinson theorem & light cones (I)

Theorem: For each finite range interaction Φ, there exists a finite group velocity V_Φ and a strictly positive increasing function μ such that for $v > V_\Phi$:

$$\lim_{|t|\to\infty;|x|>v|t|} \exp(\mu(v)|t|)\|\left[\tau_t^\Phi \tau_x(A), B\right]\| = 0 \qquad (1)$$

τ_t^Φ is a one-parameter group of automorphisms defined by the interaction Φ. The Hamiltonian of a finite system Λ is $H_\Phi(\Lambda) = \sum_{X\subset\Lambda} \Phi(X)$ for strictly local A and B.
Conditions:

- $\Phi(X)$ is Hermitian far all X in $\mathbb{Z}^\nu$
- $\Phi(X + a) = \tau_a \Phi(X)$ for X in $\mathbb{Z}^\nu$ and a in $\mathbb{Z}^\nu$
- The union R_Φ of all X such that 0 is in X and $\Phi(X) \neq 0$ is a finite subset of $\mathbb{Z}^\nu$

Prerequisites: Lieb-Robinson theorem & light cones (II)

Important note: in this work

Dynamical critical exponent $\equiv$ shape of light cone

Reasoning:

- Dynamical critical exponent

$\sim$ space-time-scaling of the first maximum of the response function $D(t, x)$

$\sim$ speed at which information can reach a given point

$\Rightarrow$ Dynamical critical exponent can be compared with group velocity V_Φ of Lieb-Robinson theorem

Prerequisites: RG treatment – A Howto

The three steps in the renormalization group transformation:

- Elimination of fast modes by scaling of the "cutoff" $\Lambda \to \Lambda/s$
- Introduction of rescaled momenta $k' \equiv sk$ that go all the way to Λ
- Rewrite effective action with rescaled fields $\Phi'(k') = \zeta \Phi_{\text{slow modes}}(k'/s)$ with a ζ chosen such that a certain coupling in the action's quadratic part remains with the same coefficient.

Quantities of interest (I)

Correlation function: $iG(t,x) = \langle \mathcal{T} \hat{\phi}(t,x) \hat{\phi}(0,0) \rangle$
Response function: $iD(t,x) = \Theta(t) \langle [\hat{\phi}(t,x), \hat{\phi}(0,0)] \rangle$

$\mathcal{T} \equiv$ time-ordering operator
$\hat{\phi} \equiv$ is the field in the Heisenberg picture (corresponding to the field theory of the model, see next slide)

$\Theta(t) \equiv$ Heaviside step function.

Quantities of interest (II)

For quadratic scalar field theories with dispersion ω_q–

- Imaginary time-ordered correlation function: $\mathcal{G}(\omega, q) = \frac{1}{\omega^2 + \omega_q^2}$
- Real-time-ordered correlation function:
  $iG(t, R) = \mathcal{G}(it + \text{sgn}(t)0^+, R)$
  and hence $iG(t \to 0, R) = \mathcal{G}(0, R) \propto \int d\omega dq \, \mathcal{G}(\omega, q) e^{iqR}$.
- Causal response function is analytic continuation
  $i\omega \longrightarrow \omega + i0^+$ of $\mathcal{G}$:
  $D(\omega, q) = -\mathcal{G}(\omega, q)\Big|_{i\omega \to \omega + i0^+} = \frac{1}{(\omega + i0^+)^2 - \omega_q^2}$.

Quantities of interest (III)

- Action of continuum field theory around critical point of standard TFIM ($H = -\sum_i \left(\sigma_i^z \sigma_{i+1}^z + g\, \sigma_i^x\right)$):
  $S = \int d\tau \int dx\ (\partial_\tau \phi)^2 + (\partial_x \phi)^2 + \varrho\phi^2 + u\phi^4$

- Action of continuum field theory around critical point of long-range TFIM ($H_\sigma = -\sum_{i\neq j} \frac{\sigma_i^z \sigma_j^z}{|i-j|^{1+\sigma}} - g \sum_i \sigma_i^x$):
  $S = \int d\omega \int dq\ (\omega^2 + \varrho + q^2 + B_\sigma |q|^\sigma)\, |\phi(\omega, q)|^2 + \int d\tau \int dx\ u\phi^4\,.$
  Note 1: Mixing of standard and Fourier representation (convenience).
  Note 2: Parameter $\rho \sim g - 1$ measures proximity to critical point.

Critical exponents summary

For the TFIM, with the dispersion: $\omega_q = \sqrt{\rho + q^2 + B_\sigma |q|^\sigma}$
Results for dynamical critical exponent z and decay of correlations θ:

	$0 < \sigma < \frac{2}{3}$	$\frac{2}{3} < \sigma < \frac{7}{4}$	$\frac{7}{4} < \sigma$
z	$\sigma/2$	$\sigma/2 + s(\sigma)\epsilon^2 + \mathcal{O}(\epsilon^3)$	1
θ	$1 - \sigma/2$	$1 - \sigma/2 + s(\sigma)\epsilon^2 + \mathcal{O}(\epsilon^3)$	1/4

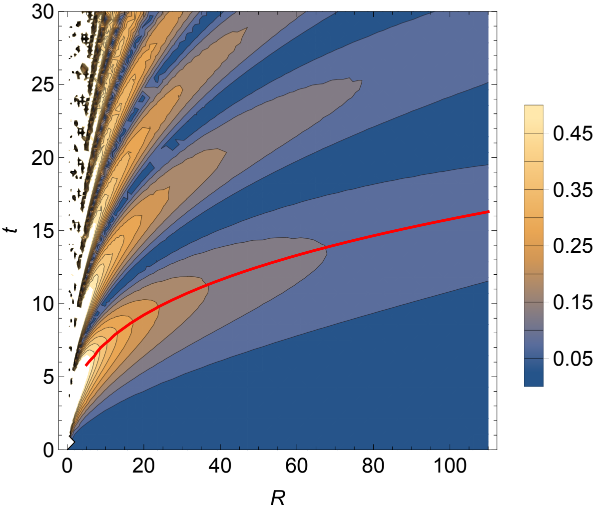
epsilon expansion: $\epsilon = 3\sigma/2 - 1$ and $s(\sigma) \approx 1/[24(1+\sigma^2)]$

Light-cones' / causal structure results summary

At criticality, the response functions D (TFIM) is described by the critical exponents z and θ, and the general scaling functions g_σ

	$0 < \sigma < 1$	$1 < \sigma < \frac{7}{4}$	$\frac{7}{4} < \sigma$
Critical	Non-linear $D(t,R) = R^{-\theta}\ g_\sigma(t/R^z)$		Linear
Non-critical	Non-linear	Linear	

Response Function: $0 < \sigma < 2$ at criticality

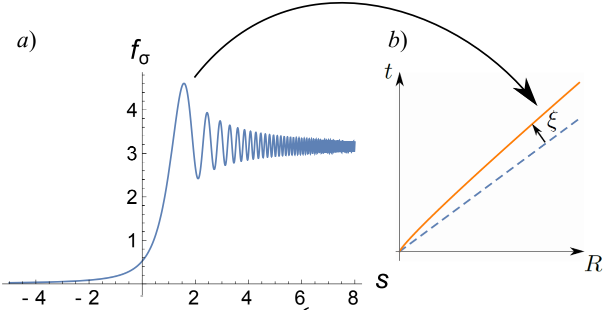


Red curve: First local maximum of $D(t, R)$ at $t \sim R^{\sigma/2}$

Plot parameter: $\sigma = \frac{2}{3}$

Response Function: $\sigma > 2$ at criticality

Dispersion $\omega_{\boldsymbol{q}} \approx |q| - |q|^{\sigma-1}$.

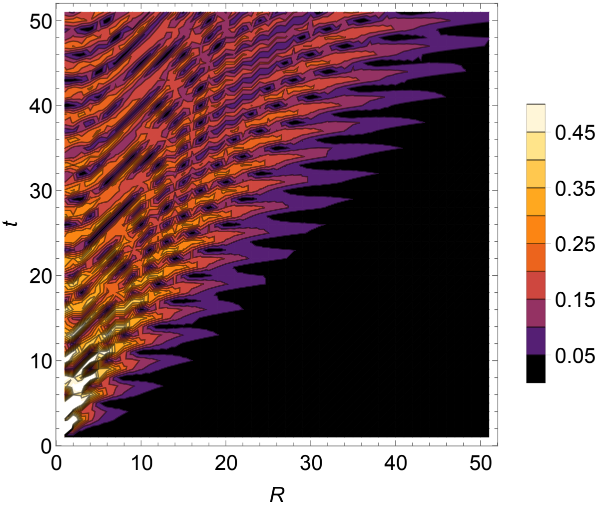


Plot parameter $\sigma = 5/2$;
$s = (t - R)/t^{1/(1-\sigma)}$
For long times $\xi/t \to 0$

$$D(t,R) \sim \int_0^\infty \frac{dq}{q} \left\{ \sin[q(t-R) - q^{\sigma-1}t] + \sin[q(t+R) - q^{\sigma-1}t] \right\}$$

$$= \tilde{f}_\sigma \left[\frac{t-R}{t^{1/(\sigma-1)}} \right] + \tilde{f}_\sigma \left[\frac{t+R}{t^{1/(\sigma-1)}} \right] \sim f_\sigma \left[\frac{t-R}{t^{1/(\sigma-1)}} \right]$$

Norm of Resp. Funct.: Any $\sigma > 0$ away from criticality



Dispersion $\omega_q = \sqrt{1 + F(q^2) + B_\sigma |q|^\sigma}$

Plot parameter: $\sigma = 1; \rho = 1$ $\omega_q = \sqrt{1 + 2q - q^2/pi}$

$$G(R) \sim \int_0^\infty dq\, e^{-qR}\, \mathrm{Im} \frac{1}{\sqrt{1+F(-q^2)+e^{i\pi\sigma/2}q^\sigma}} \sim \int_0^\infty dq\, e^{-qR}\, q^\sigma \sim \frac{1}{R^{1+\sigma}}$$

RG treatment I

- Anomalous field dimension: $\eta_{\mathrm{SR}} = \frac{1}{4}$
- For $\sigma > \frac{7}{4} = 2 - \eta_{\mathrm{SR}}$: Universal properties & critical exponents from Φ^4 model in one higher dimension
- Long-range coupling $|q|^\sigma |\phi(\omega, q)|^2$ can be considered as a perturbation to the short-range TFIM ($[\Phi] = \eta_{\mathrm{SR}}/2$) as long as $\sigma < 2 - \eta_{\mathrm{SR}}$.
- First consider $d-$dimensional action for $\sigma < 2 - \eta_{\mathrm{SR}}$:
  $S = \int d\tau \int d^d\mathbf{x} \ \left[A\left(\partial_\tau \phi\right)^2 + \varrho\phi^2\right]$
  $+B_\sigma \int d\tau \int d^d\mathbf{x} \int d^d\mathbf{y} \frac{\phi(\tau,\mathbf{x})\phi(\tau,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+\sigma}} + u \int d\tau \int d^d\mathbf{x} \ \phi^4$

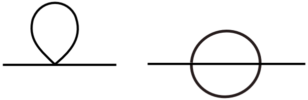
RG treatment II

$$S = \int d\tau \int d^d\mathbf{x} \; \left[A\left(\partial_\tau \phi\right)^2 + \varrho\phi^2\right]$$
$$+B_\sigma \int d\tau \int d^d\mathbf{x} \int d^d\mathbf{y} \frac{\phi(\tau,\mathbf{x})\phi(\tau,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+\sigma}} + u \int d\tau \int d^d\mathbf{x} \; \phi^4$$

- Rescaling: $\mathbf{x} \to \mathbf{x}' = \mathbf{x}/b, \quad \tau \to \tau' = \tau/b^z, \quad \phi \to \phi' = b^a\phi$
- Dimensions: $[A] = -z + d - 2a, [B_\sigma] = z - \sigma + d - 2a,$ $[\varrho] = z + d - 2a, [u] = z + d - 4a$
- Mean-field: quadratic part of S (except for ϱ) is invariant under scaling $\to [A] = [B_\sigma] = 0$; thus: $z = \sigma/2$ and $\eta = 1 - \sigma/2$.
- Mean-field-anomalous field dimension from $a = (d - 1 + \eta)/2$
- Mean-field upper bound for relevance of coupling: $[u] = 0 \to d_u = 3/2\sigma \to$ one-dimensional quadratic model fully captures physics for $\sigma < 2/3$.

RG treatment III

- Beyond mean-field terms for $\sigma \geq 2/3$: Corrections from expansion in $\epsilon = d_u - d = 3/2\sigma - 1$.
- Note: RG procedure may not renormalize non-analytical contributions to the action, hence $[B_\sigma] = 0$ and $\eta = 1 + z - \sigma$ to all orders of perturbation theory.
- ϵ-expansion in loop diagrams for two-point function:



yields: $\Delta\eta = \Delta z = s(\sigma)\epsilon^2 + \mathcal{O}(\epsilon^3)$ where $s(\sigma) \approx 1/[24(1+\sigma^2)]$.[1]

[1] $s(\sigma) = \frac{4\Gamma(3\sigma/2)}{27\sqrt{\pi}\Gamma(3\sigma/4-1/2)} \int_0^\infty dy \int_0^\pi d\theta \, \frac{y^{\sigma-1}(\sin\theta)^{3\sigma/2-2}}{(1+y^2+2y\cos\theta)^{\sigma/4}\left[1+y^{\sigma/2}+(1+y^2+2y\cos\theta)^{\sigma/4}\right]^3}$.