Phase diagram of the interacting Majorana chain

arXiv:1505.03966

Emergent Supersymmetry from Strongly Interacting Majorana Zero Modes

arXiv:1504.05192

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Outline

- **1D models of strongly correlated electrons**
- **Physical Realization**
- **Phase Diagram**
  - Strong and weak coupling limits
  - Attractive interactions
  - Repulsive interactions
- **Experimental Signatures**

1D models of strongly correlated electrons

- (a) Hubbard chain:

$$H = -t\sum_{j\sigma}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + H.C.) + U\sum_{j}\hat{n}_{j,\uparrow}\hat{n}_{j,\downarrow} \quad \hat{n}_{j,\sigma} = c^{\dagger}_{j,\sigma}c_{j,\sigma}$$

- (b) Dirac chain (spinless fermions)

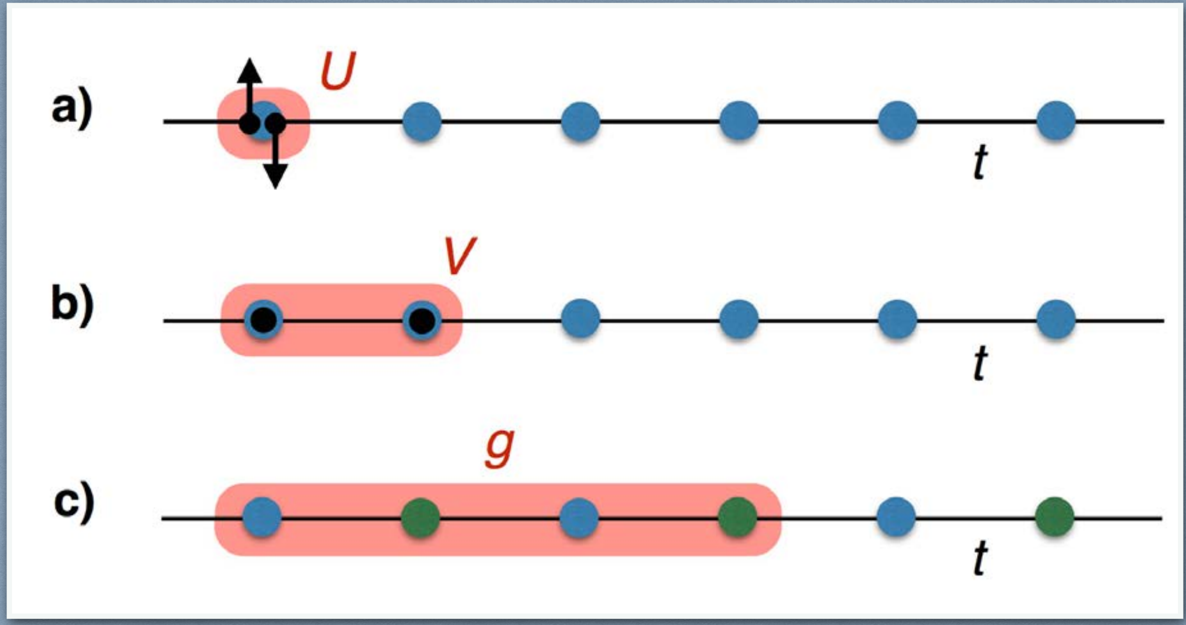
$$H = \sum_{j}\left[-t(c^{\dagger}_{j}c_{j+1} + H.C.) + V(\hat{n}_{j} - 1/2)(\hat{n}_{j+1} - 1/2)\right]$$

- (c) Majorana chain

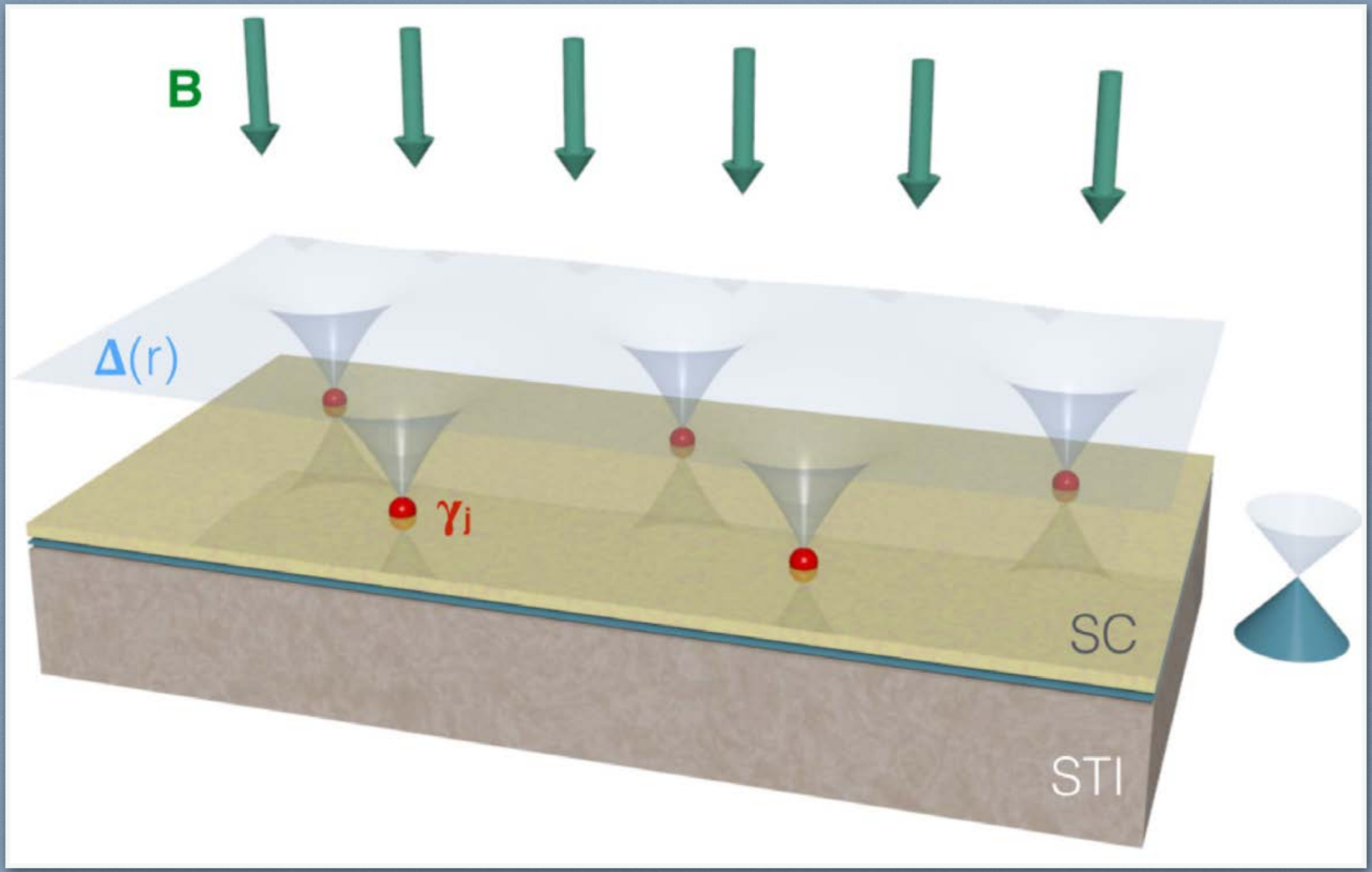
$$H = \sum_{j}[it\gamma_j\gamma_{j+1} + g\gamma_j\gamma_{j+1}\gamma_{j+2}\gamma_{j+3}]$$

$$\gamma^{\dagger}_{j} = \gamma_j, \qquad \{\gamma_j, \gamma_i\} = 2\delta_{ij}$$

$$\gamma^{2}_{j} = 1$$



Physical Realization



Superconducting order is induced in the surface of a strong topological insulator (STI). Magnetic field **B** induces Abrikosov vortices in the SC order parameter. Each vortex hosts an unpaired Majorana zero mode γ_j.

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \qquad \gamma_i^\dagger = \gamma_i$$

C.K. Chiu, D. I. Pikulin, and M. Franz, Phys. Rev. B ***91****, 165402 (2015)*

Liang Fu and C. L. Kane, Phys. Rev. Lett. ***100*** *096407 (2008)*

Physical Realization

Interaction term: $$H_{\text{int}} = \sum_{ijkl} g_{ijkl}\gamma_i\gamma_j\gamma_k\gamma_l$$

↑ ***real constants***

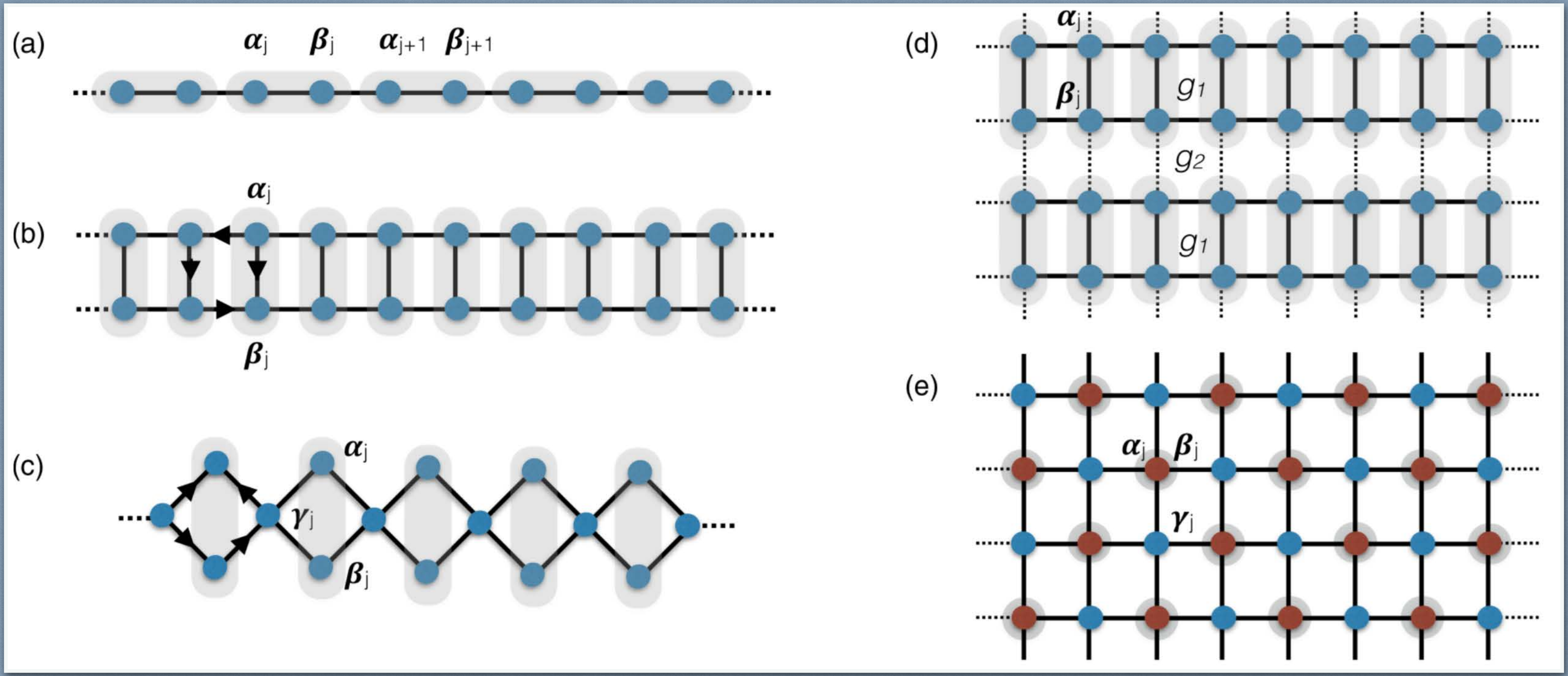
The interaction term arises from Coulomb interactions

Interaction strength $$g \approx (10.6\ \text{meV}) \times e^{-d^2/\pi\xi^2}$$

Hopping amplitude $$t \approx \mu e^{-d^2/4\pi\xi^2}$$

Access the strong coupling regime $|t| \ll |g|$ by tuning μ

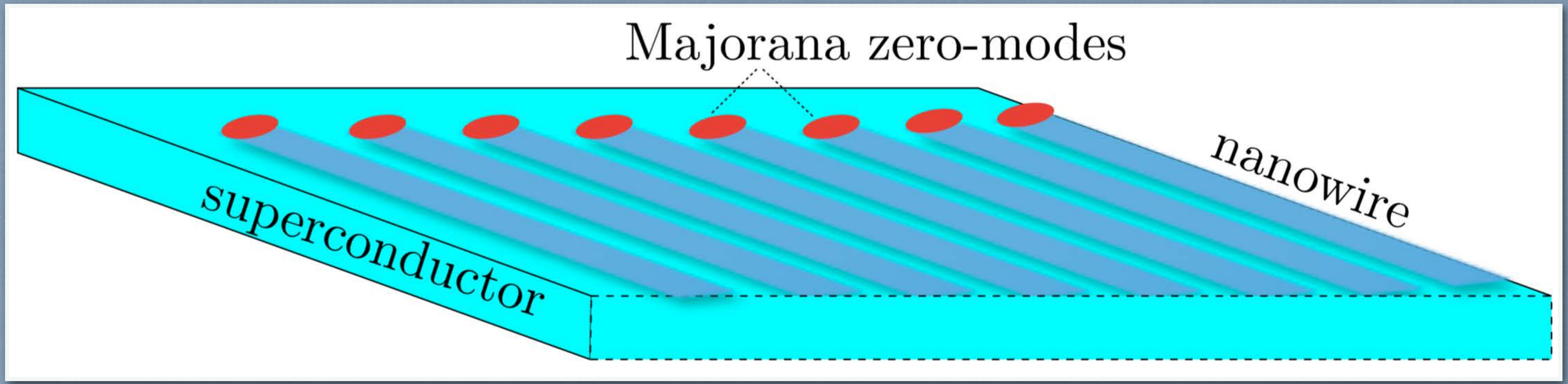
Lattice structures for interacting Majorana fermions



a) 1D chain
b) two-leg ladder
c) diamond chain
d) square lattice
e) modified square lattice with with alternate sites occupied by double vortices

Lattice structures for interacting Majorana fermions

A. Milsted, L. Seabra, I. C. Fulga, C. W. J. Beenakker, E. Cobanera,
Phys. Rev. B **92**, 085139 (2015)



Array of nanowires on a superconducting substrate, with a delocalized Majorana edge mode composed out of coupled zero-modes localized at the end points.

The end points of each nanowire form a 1D lattice of Majorana operators

$$H = -i \sum_s \alpha_s \gamma_s \gamma_{s+1} - \sum_s k_s \gamma_s \gamma_{s+1} \gamma_{s+2} \gamma_{s+3}$$

Phase diagram

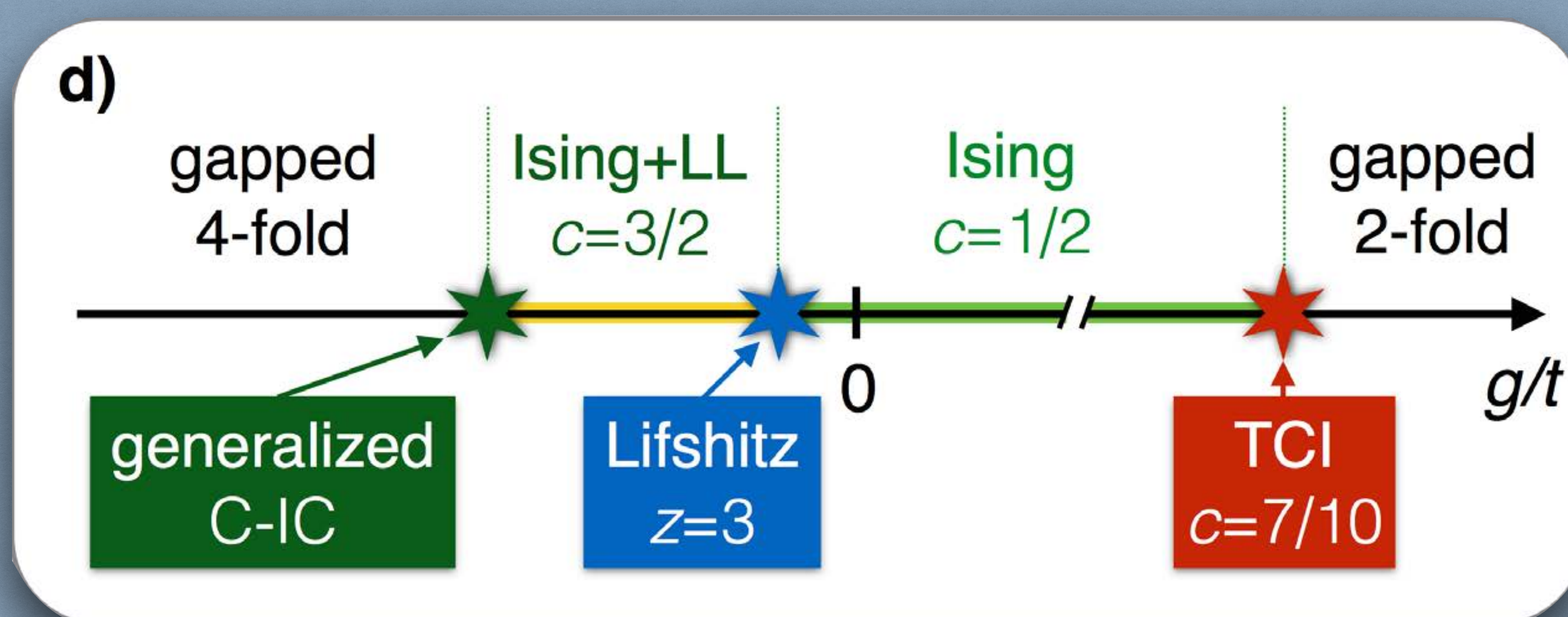
$$H = \sum_j \left[it\gamma_j\gamma_{j+1} + g\gamma_j\gamma_{j+1}\gamma_{j+2}\gamma_{j+3} \right]$$

Dirac fermion operators: $c_j = (\gamma_{2j} + i\gamma_{2j+1})/2$

$$\hat{p}_j = 2\hat{n}_j - 1$$

$$H = \sum_j \left(t \left[\hat{p}_j - (c_j^\dagger - c_j)(c_{j+1}^\dagger - c_{j+1}) \right] + g \left[-\hat{p}_j\hat{p}_{j+1} + (c_j^\dagger - c_j)\hat{p}_{j+1}(c_{j+1}^\dagger + c_{j+2} \right] \right)$$

Solved using DMRG and field theory/RG considerations



Strong coupling limit

Nontrivial ground state!

Consider a model with alternating hopping and interaction terms:

$$H = \sum_j (it\gamma_{2j}\gamma_{2j+1} + it_2\gamma_{2j+1}\gamma_{2j+2} + g_1\gamma_{2j}\gamma_{2j+1}\gamma_{2j+2}\gamma_{2j+3} + g_2\gamma_{2j+1}\gamma_{2j+2}\gamma_{2j+3}\gamma_{2j+4})$$

↓ ***Jordan-Wigner transformation***

$$H = t_1 \sum_j \sigma_j^z - t_2 \sum_j \sigma_j^x \sigma_{j+1}^x - g_1 \sum_j \sigma_j^z \sigma_{j+1}^z - g_2 \sum_j \sigma_j^x \sigma_{j+2}^x$$

Simplicity arises when g_2=0 and t_1=0=t_2

Combine every second pair of Majorana's to make a Dirac

$$c_j = (\gamma_{2j} + i\gamma_{2j+1})/2$$

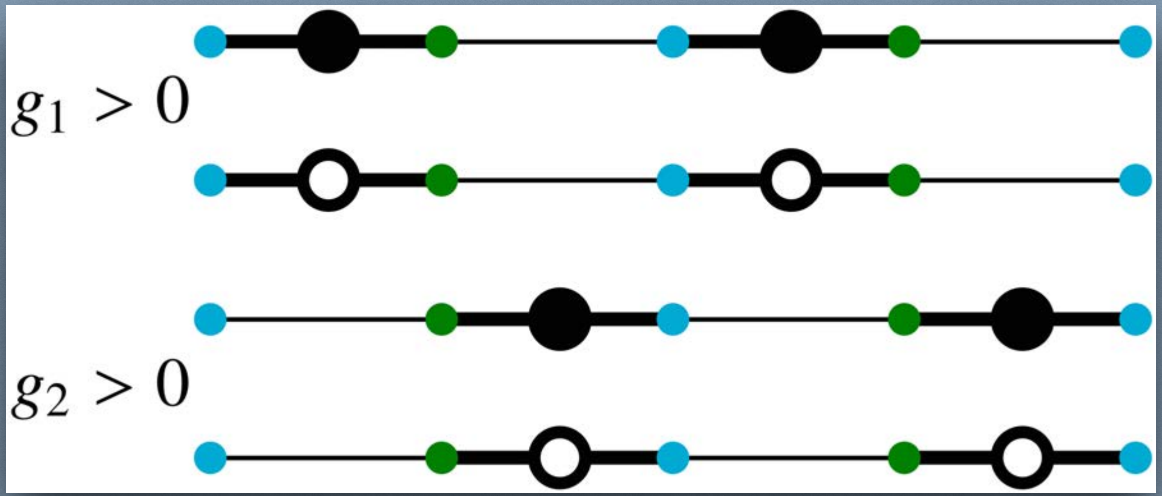
$$i\gamma_{2j}\gamma_{2j+1} = 2c_j^\dagger c_j - 1 = 2n_j - 1$$

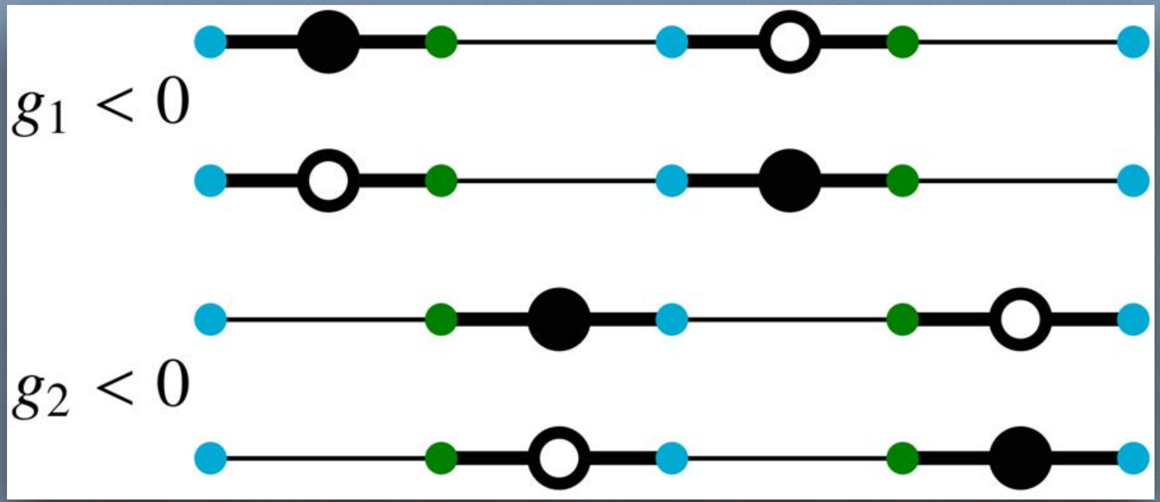
$$H = -g_1 \sum_j (2\hat{n}_j - 1)(2\hat{n}_{j+1} - 1) = -g_1 \sum_j \hat{p}_j \hat{p}_{j+1}$$

Strong coupling limit

Attractive Interactions g>0: $p_j = 1$ or $p_j = -1$ FM Ising chain

Repulsive Interactions g<0: $p_j = \pm(-1)^j$ AFM Ising chain





Sites of the Majorana chain

 Occupied Dirac level  Unoccupied Dirac level

Strong coupling limit

Small nonzero t in the dimerised Hamiltonian:

$$H = \sum_j (t\hat{p}_j - g\hat{p}_j\hat{p}_{j+1})$$

The effects of the hopping term are different and depend on the sign of g

Attractive Interactions g>0

- t>0 Empty Dirac levels
- ↕ ***First order transition with a jump in $\langle \hat{p}_j \rangle$ at t=0***
- t<0 Filled Dirac levels

Repulsive Interactions g<0

Degenerate ground states for $|t_1| < |g_1|$

There is a critical t above which the ground state has either all levels empty or filled, depending on the sign of t.

Spin chain representation $\hat{p}_j = \sigma_j^z$

Weak coupling limit

Low energy excitations with linear dispersion of slope ν=4t at k=0 and k=π

$$\gamma_j(t) = 2\gamma_R(\nu t - j) + (-1)^j 2\gamma_L(\nu t + j)$$

$\gamma_{R/L}(\nu t \mp j)$ Relativistic right/left moving Majorana fermion field

Noninteracting limit:

$$H_0 = i\nu \int dx[\gamma_R \partial_x \gamma_R - \gamma_L \partial_x \gamma_L]$$

→ Massless conformal field theory with c=1/2, corresponding to the transverse field Ising model

Interaction term:

$$H_{int} \approx -256g \int dx \gamma_R \partial_x \gamma_R \gamma_L \partial_x \gamma_L$$

→ RG scaling = 4

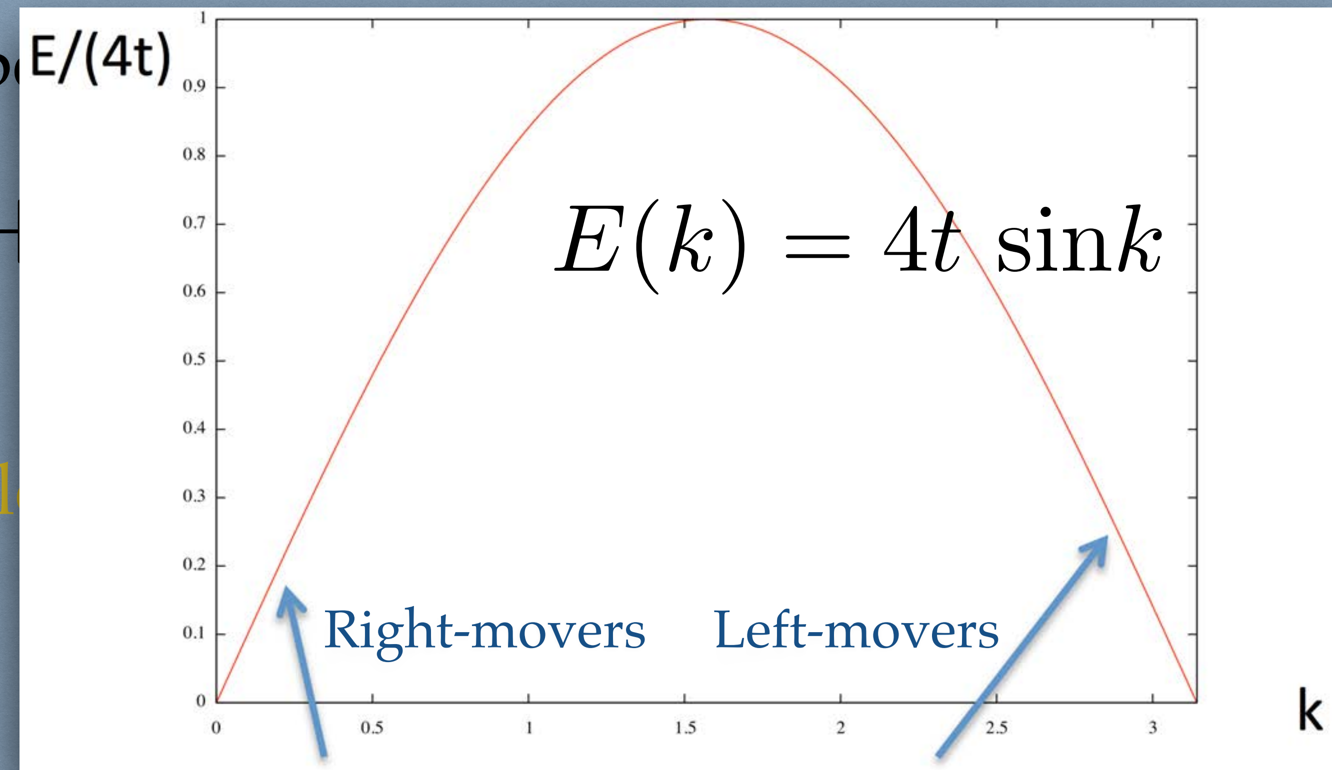
Extended massless Ising phase in the vicinity of g=0

Weak coupling limit

Low energy excitations with linear disp

$$\gamma_j(t) = 2\gamma_R(\nu t - j) +$$

$\gamma_{R/L}(\nu t \mp j)$ Relativistic right/l



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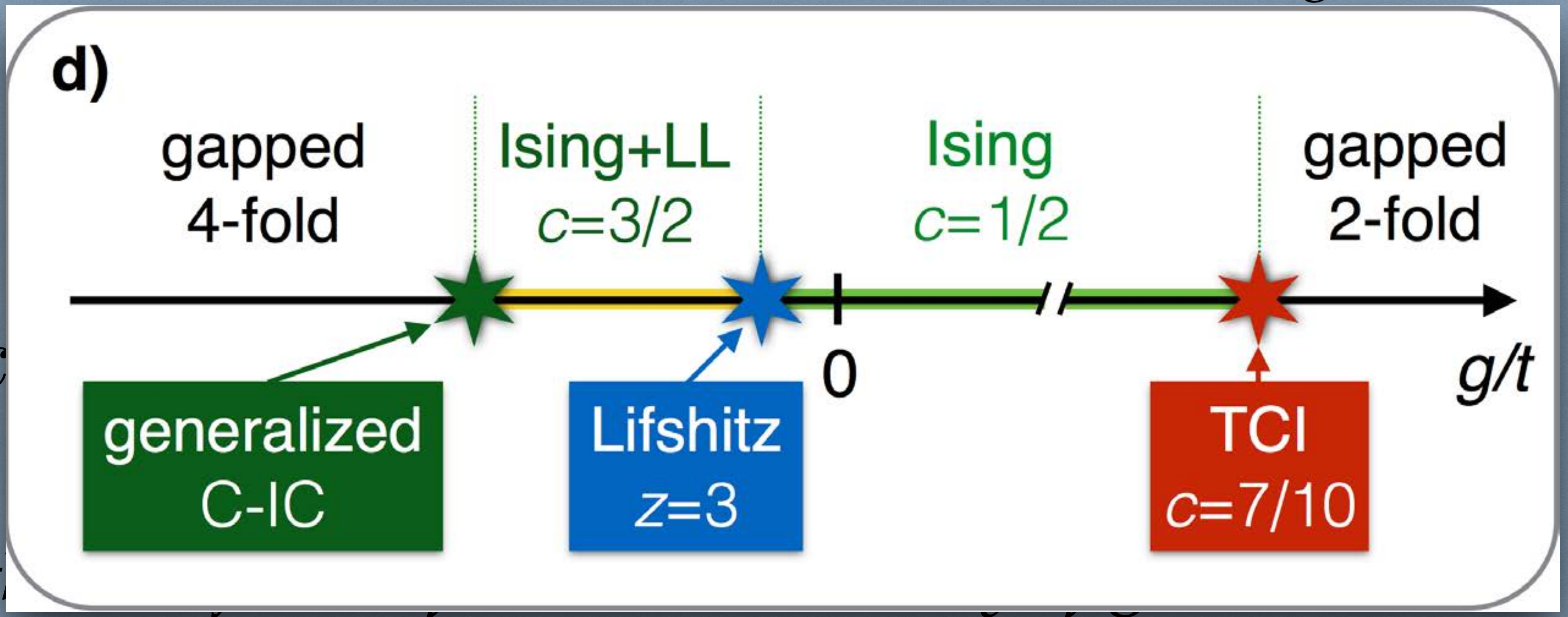
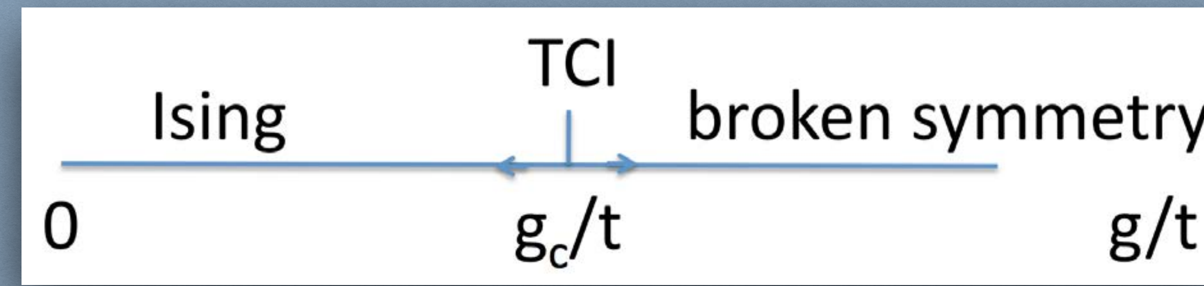


Extended phase wit

Attractive Interactions

The second simplest minimal model is the TriCritical Ising model with c=7/10



Fermion parity for a chain of L=2l

$$F = \prod_{j=1}^{l-1} (i\gamma_{2j}\gamma_{2j+1}) = \prod_{j=1}^{l-1} \hat{p}_j = (-1)^{N_F + l}$$

Number of Dirac fermions added to the vacuum state

E_0 ground state energy

E_1 first excited state energy

E_A Anti periodic boundary conditions

E_P Periodic boundary conditions

E^{odd} Odd fermion parity sector

E^{even} Even fermion parity sector

ϵ_0 energy density of the ground state in the thermodynamic limit

Universal ratios at the critical point

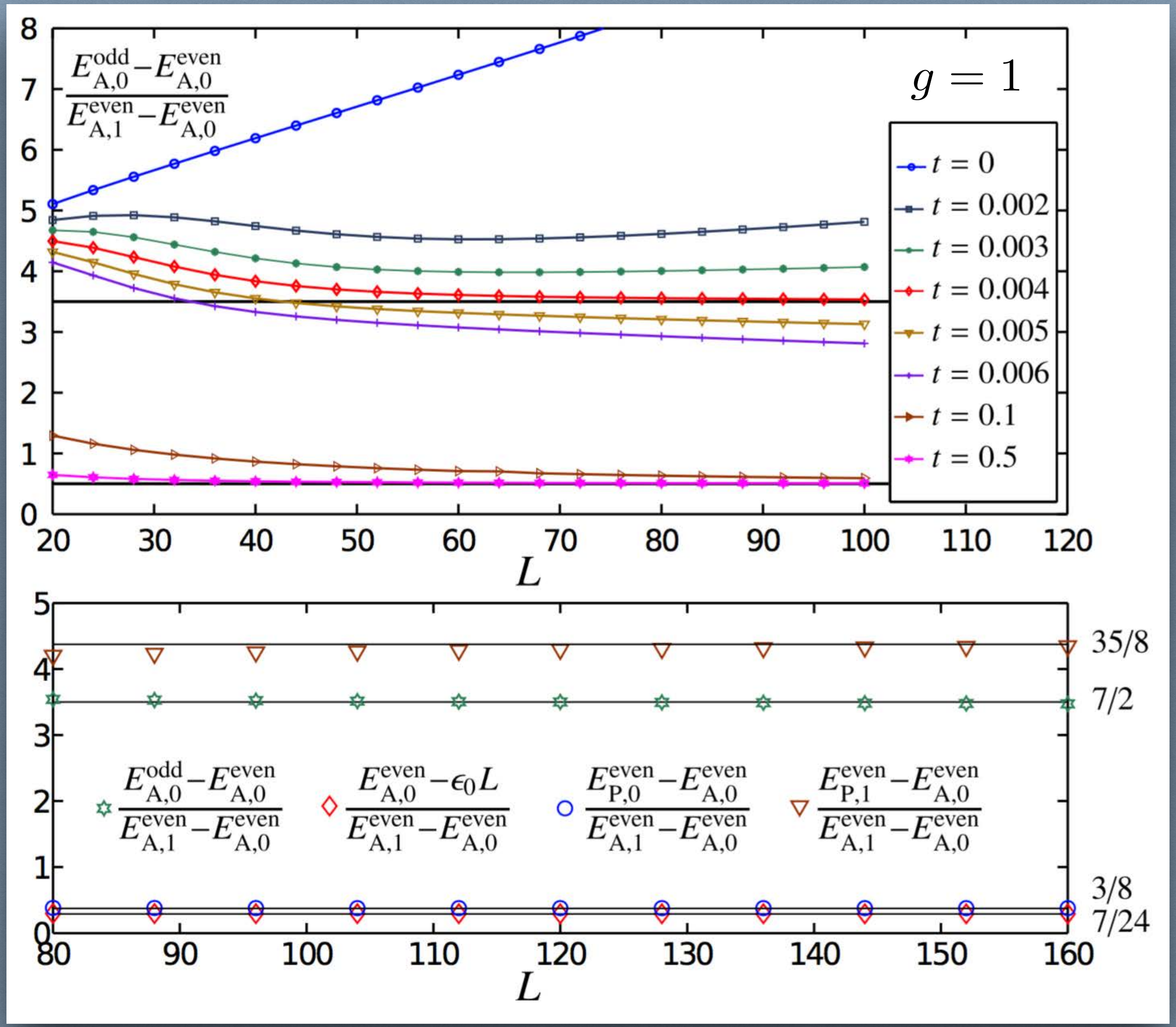
CFT	c	$\frac{E^{\text{odd}}_{\text{A},0} - E^{\text{even}}_{\text{A},0}}{E^{\text{even}}_{\text{A},1} - E^{\text{even}}_{\text{A},0}}$	$\frac{E^{\text{even}}_{\text{P},0} - E^{\text{even}}_{\text{A},0}}{E^{\text{even}}_{\text{A},1} - E^{\text{even}}_{\text{A},0}}$	$\frac{E^{\text{even}}_{\text{P},1} - E^{\text{even}}_{\text{A},0}}{E^{\text{even}}_{\text{A},1} - E^{\text{even}}_{\text{A},0}}$	$\frac{E^{\text{even}}_{\text{A},0} - \epsilon_0 L}{E^{\text{even}}_{\text{A},1} - E^{\text{even}}_{\text{A},0}}$
Ising	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
TCI	$\frac{7}{10}$	$\frac{7}{2}$	$\frac{3}{8}$	$\frac{35}{8}$	$\frac{7}{24}$

Attractive Interactions

$$E^{\text{even}}_{A,0} = \epsilon_0 L - \frac{2\pi\nu}{L}\frac{c}{12}$$ → Direct access to central charge

Detect the $t_c = 0.00405$

Use t_c to test all ratios



The Tricritical Ising model is the only conformal field theory that exhibits supersymmetry

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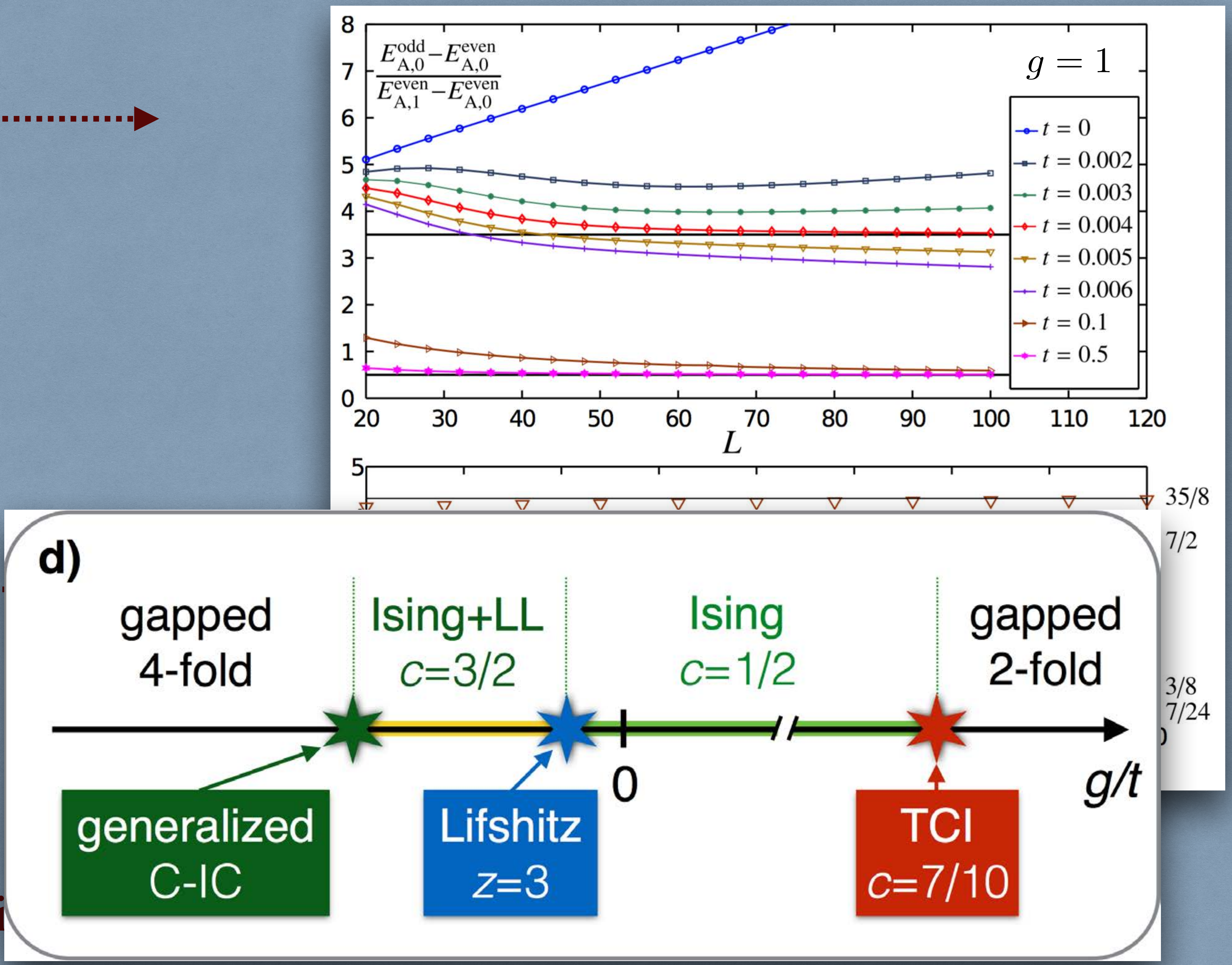


The Tricritical Ising model i
supersymmetry

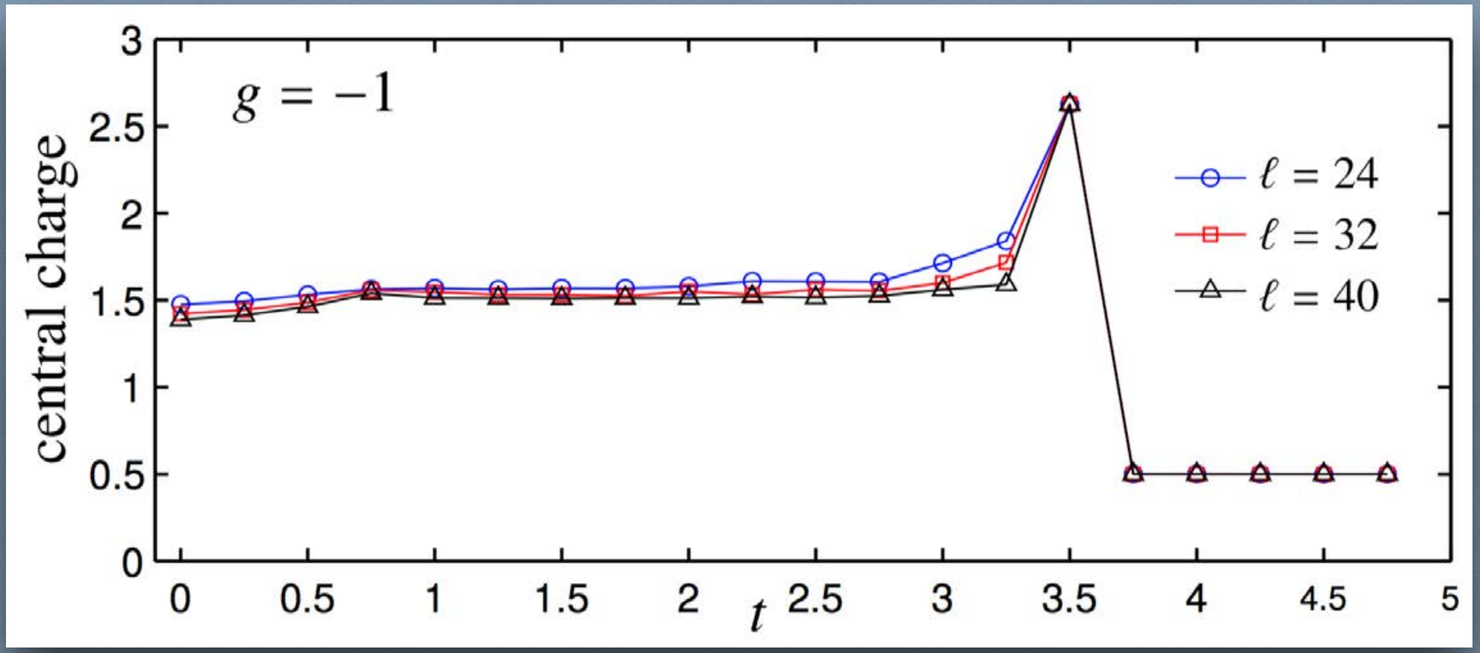
Repulsive Interactions g<0

Calculate **central charge** from DMRG calculations of the entanglement entropy

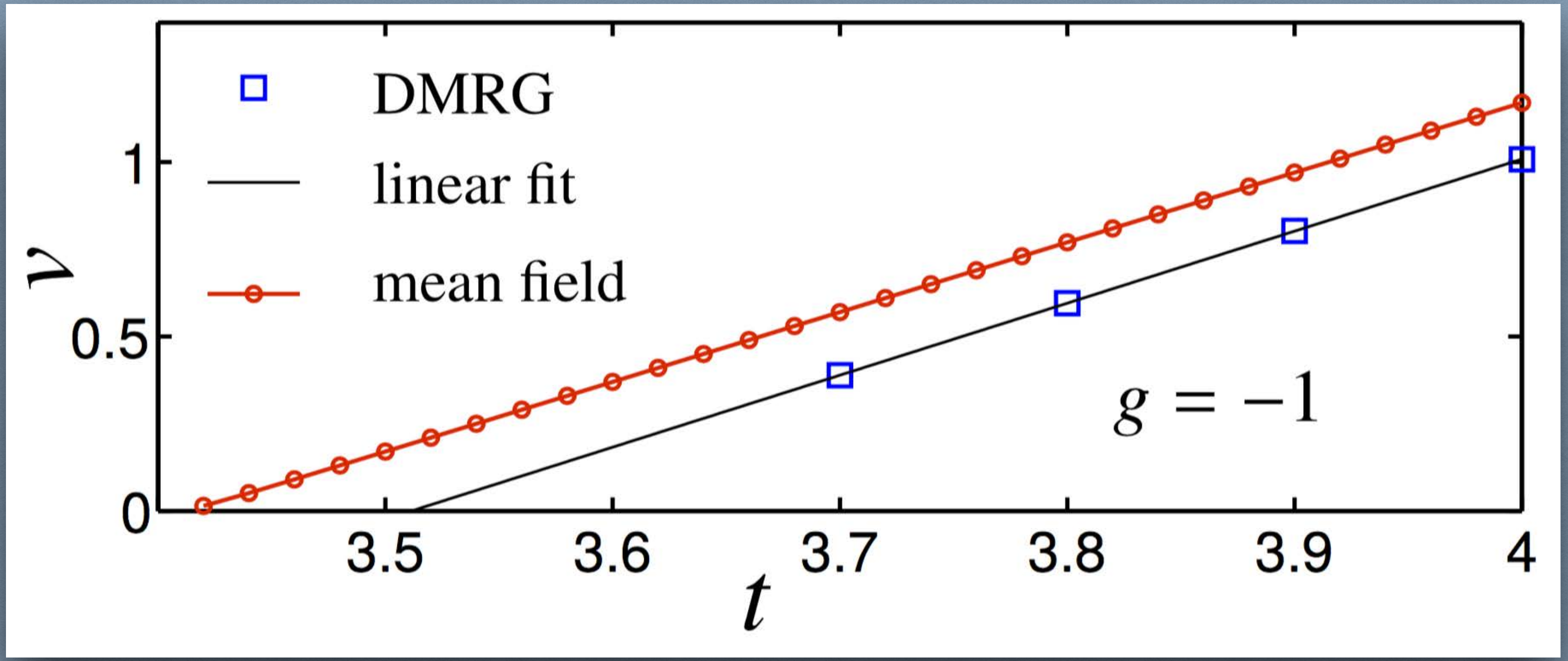
Lifshitz transition

t/g= -3.512

g/t=-0.285



c = 3/2 ⇒ 3 species of low energy Majoranas



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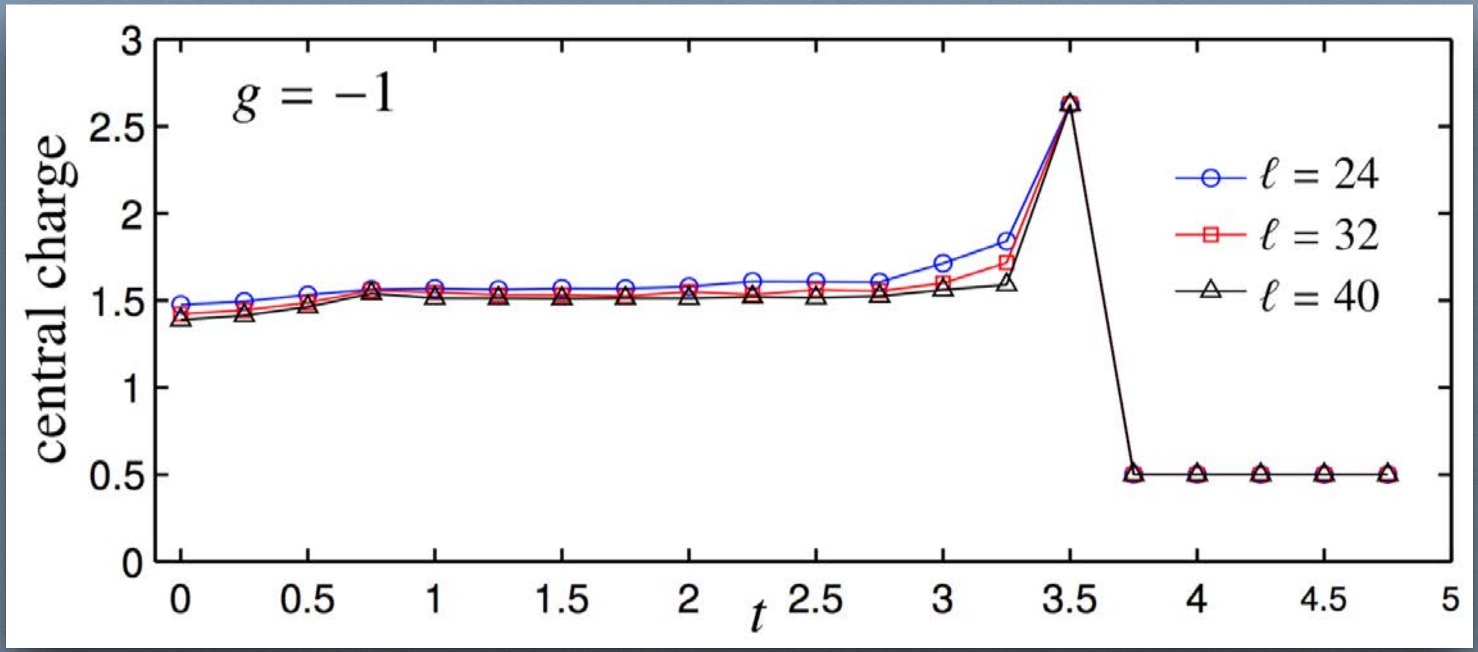
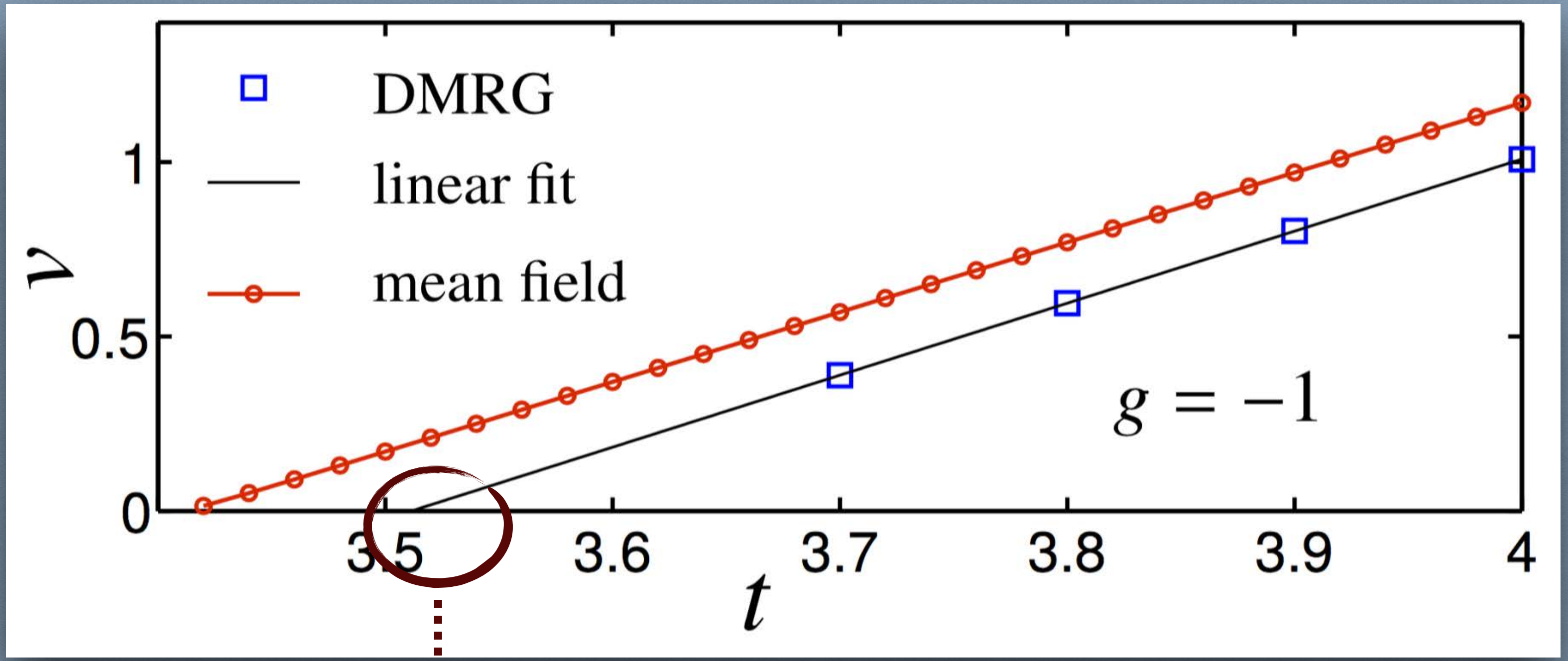
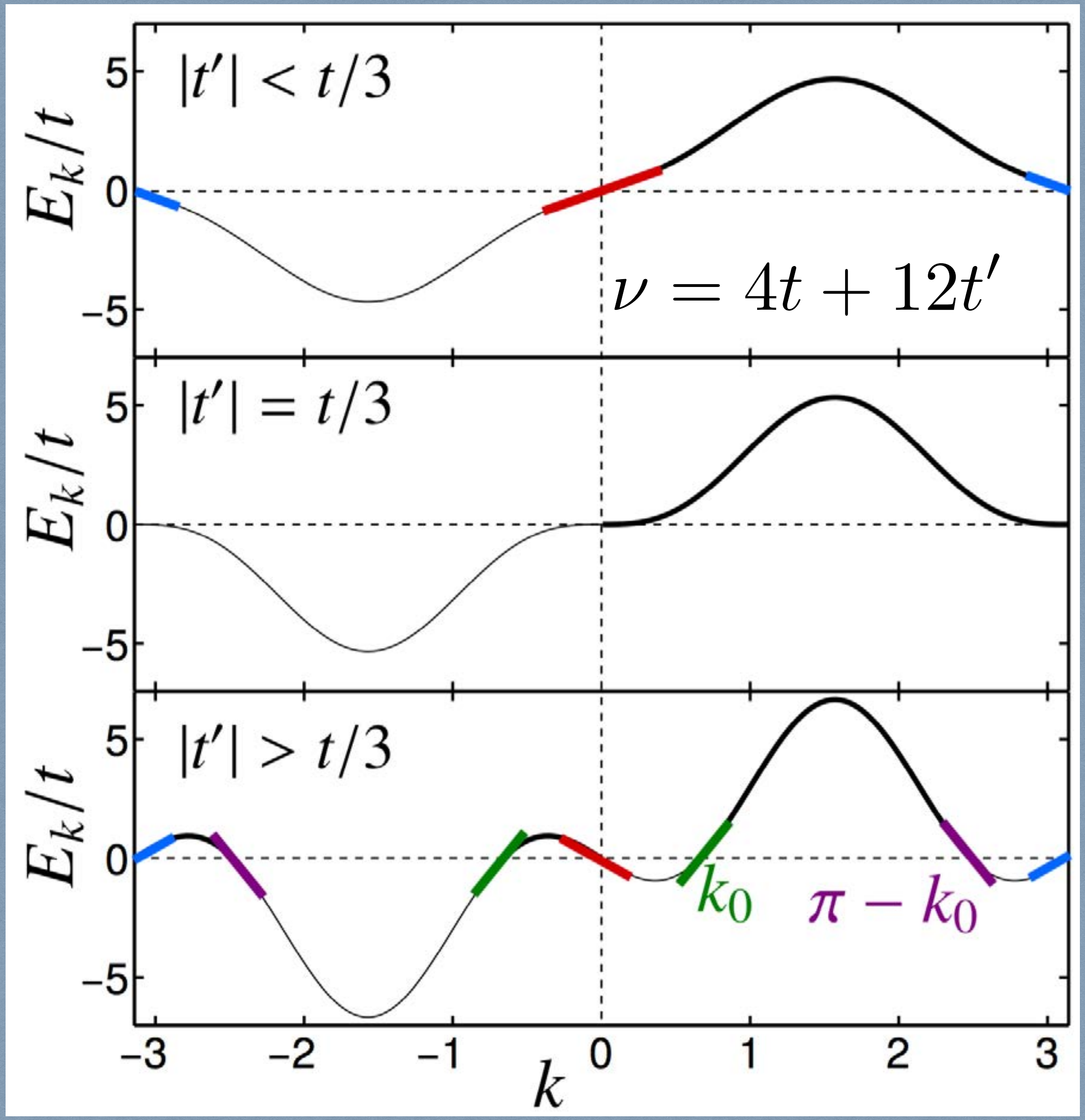


$c = 3/2 \Rightarrow$ 3 species of low energy Majoranas



same behaviour with a model with third neighbour hopping

Repulsive Interactions g<0



$$H = i\sum_j \gamma_j[t\gamma_{j+1} + t'\gamma_{j+3}] = \frac{1}{2}\sum_k E_k\gamma(-k)\gamma(k)$$

$$E_k = 4t\sin k + 4t'\sin 3k$$

$$t > 0\ ,\ t' < 0$$

$$\sin k_0 = \frac{1}{2}\sqrt{3 + t/t'}$$

Velocity at k=0 $\quad \nu_0 = 16\sin^2 k_0$

Velocity at k=k_0 $\quad \nu = 2\nu_0\cos k_0$

Majorana and Dirac fermion with left/right movers

$$\gamma_j \approx 2\gamma_L(j) + (-1)^j 2\gamma_R(j) + [e^{-ik_0 j}\psi_R(j) + e^{i(k_0-\pi)}\psi_L + \text{H.C.}]$$

$$H_0 = i\int dx\left(\nu_0(\gamma_R\partial_x\gamma_R - \gamma_L\partial_x\gamma_L) + \nu(\psi_R^\dagger\partial_x\psi_R - \psi_L^\dagger\partial_x\psi_L)\right)$$

Repulsive Interactions g<0

Effect of interactions

1.
$$H_{\text{int}} \approx \int dx g_0 : \psi_L^\dagger \psi_L \psi_R^\dagger \psi_R :$$

$$g_0 = -16g(\cos k_0 - \cos 3k_0) > 0$$

→ Luttinger Liquid

$$K = 1 - \frac{g_0}{2\pi\nu} < 1$$

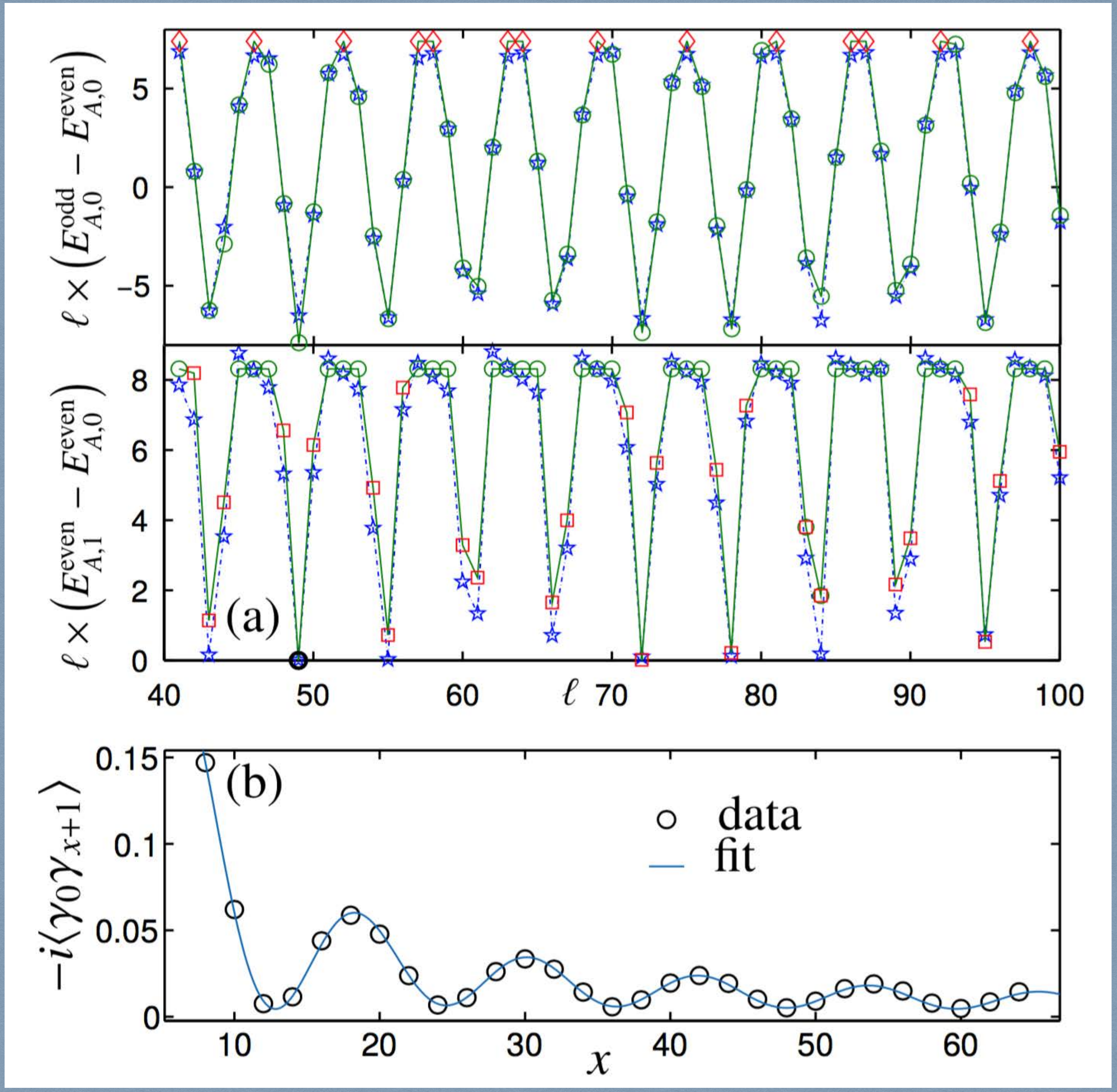
2.
$$H_{\text{int}} \approx g_1 \int dx \gamma_R \gamma_L (\psi_R \psi_L + \psi_R^\dagger \psi_L^\dagger)$$

→ RG scaling > 2
Irrelevant!

Ising+LL phase (Free Majorana particles)

Repulsive Interactions g<0

Comparison with DMRG results



g=-1
t=2.25
☆DMRG data
◯ LL excitations
◇ Ising excitations

Fitting parameters

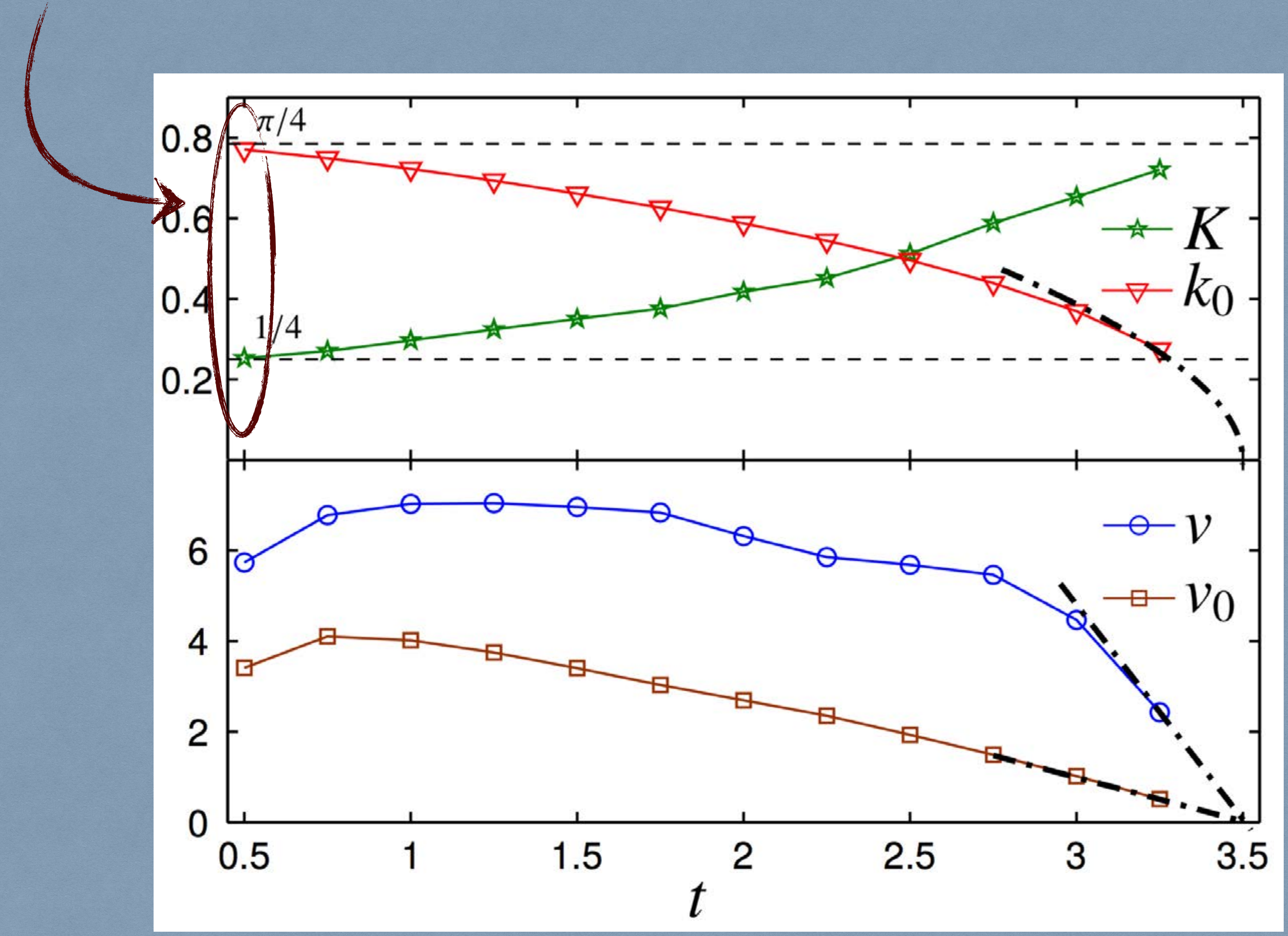
$$K = 0.4517\,,\ \ k_0 = 0.5444$$

Fitting parameters

$$K = 0.4611\,,\ \ k_0 = 0.5373$$

Repulsive Interactions g<0

$$H' \propto \int dx \gamma_R \gamma_L [e^{i(4k_0-\pi)} \psi_R^\dagger \partial_x \psi_R^\dagger \psi_L \partial_x \psi_L - \text{H.C.}]$$ with RG scaling < 2



Repulsive Interactions g<0

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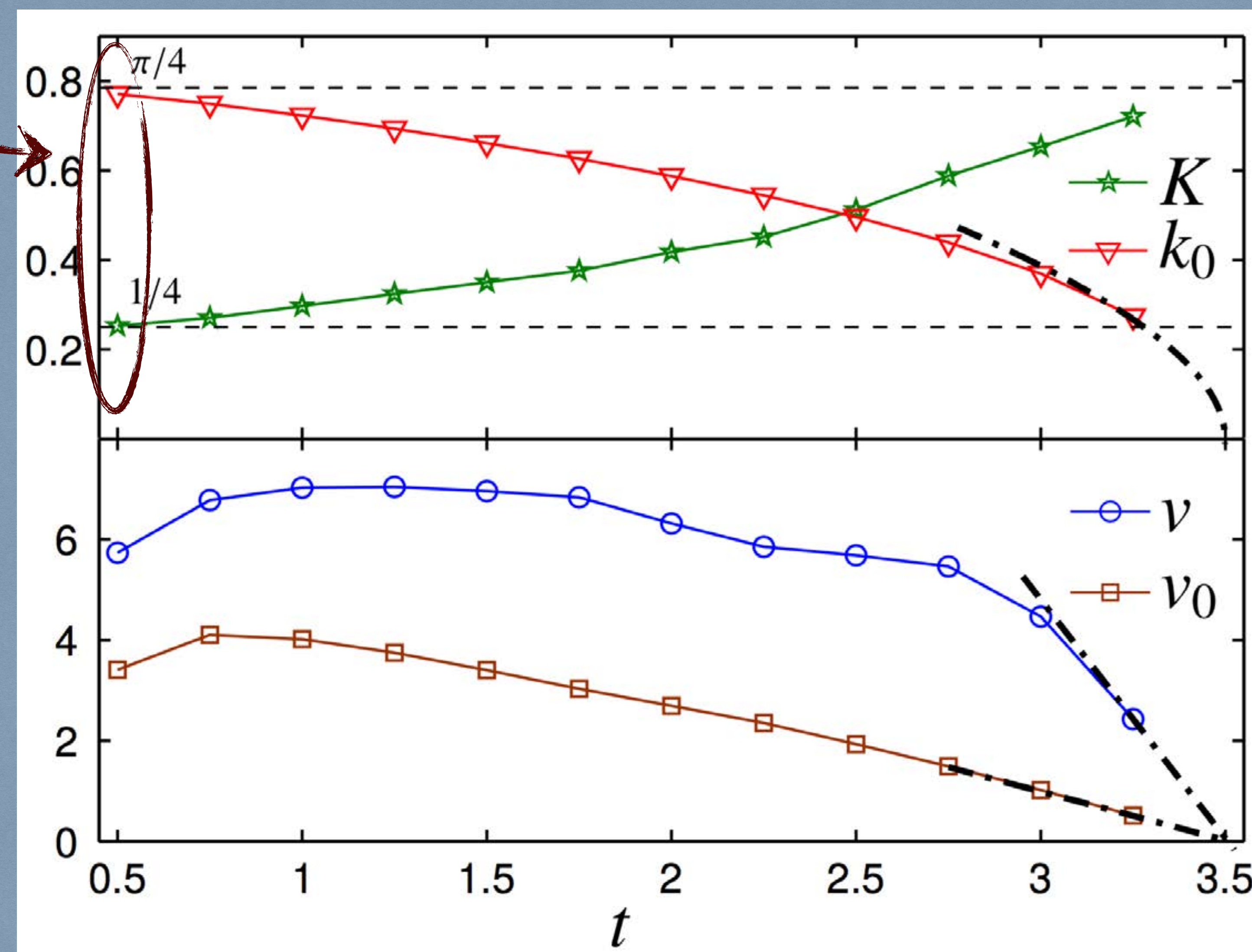

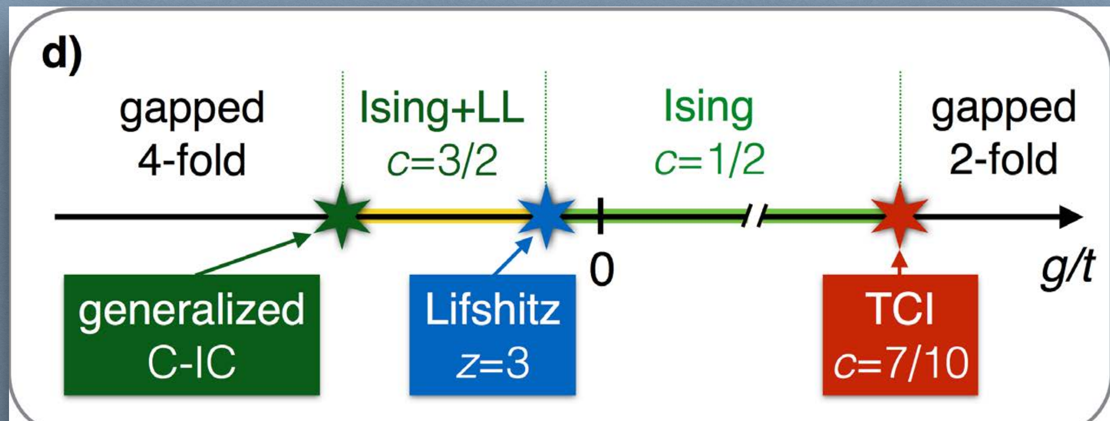

Critical value for the C-IC transition

$$g/t = -2.86$$

Experimental Signatures

Experimental signatures using scanning tunnelling microscopy (STM)

Tunneling current $\langle I \rangle \propto \text{Im} G_R(-eV)$

$$G_R(\omega) = -i \int dt e^{i\omega\ t} \langle [\gamma_j(t)\psi_0(t), \gamma_j(0)\psi_0^\dagger(0)] \rangle$$

ψ_0 Annihilates an electron at the tip

$$I_{\text{Ising}} \propto V$$

$$I_{\text{TCI}} \propto \text{sign}(V)|V|^{7/5}$$

$$I_{\text{Lifshitz}} \propto |V|^{1/3}$$

$$I_{\text{LL}} \propto V^{(K+1/K)/2}$$